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Integrate-and-fire models with an almost periodic input function

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Abstract

We investigate leaky integrate-and-fire models (LIF models for short) driven by Stepanov and μ-almost periodic functions. Special attention is paid to the properties of the firing map and its displacement, which give information about the spiking behavior of the considered system. We provide conditions under which such maps are well-defined and are uniformly continuous. We show that the LIF models with Stepanov almost periodic inputs have uniformly almost periodic displacements. We also show that in the case of μ-almost periodic drives it may happen that the displacement map is uniformly continuous, but is not μ-almost periodic (and thus cannot be Stepanov or uniformly almost periodic). By allowing discontinuous inputs, we extend some previous results, showing, for example, that the firing rate for the LIF models with Stepanov almost periodic input exists and is unique. This is a starting point for the investigation of the dynamics of almost-periodically driven integrate-and-fire systems.

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1. Introduction

Integrate-and-fire models are commonly used for modeling the activity of neuronal cells (see for example [17,20,22,37]). Although they are not able to capture all the electrophysiological phenomena, as a part of a big neural network, they are computationally more efficient than, for example, the classical Hodgkin–Huxley model (see for example [17,19]), and their biological relevance is in some cases satisfactory. In particular, the so-called *leaky integrate-and-fire model*[1]

$$\dot{x}(t) = -\sigma x(t) + f(t) \qquad \text{for a.e. } t \in \mathbb{R}, \tag{1}$$

where $\sigma \geq 0$, is one of the models in the center of interest of neuroscientists. This model dates back to Lapicque (see [25]), who discovered that the voltage x across the cell membrane decays exponentially to its resting state x_r and only an external input f, which might be current injected via an electrode or the impulse from a pre-synaptic neuron, might cause the increase of the voltage. Spiking (or firing), that is emitting an action potential, is introduced to this simple dynamics by adding the resetting condition

$$x(s) = x_\vartheta \quad \Longrightarrow \quad \lim_{t \to s^+} x(t) = x_r, \tag{2}$$

which says that after the dynamical variable reaches a certain threshold x_ϑ at some time s, it is immediately reset to its resting value x_r and the dynamics continues again from the point (t, x_r). Although usually constant thresholds and resets are studied, it is also possible to consider integrate-and-fire models with varying bounds, allowing thus to incorporate into the model additional biological phenomena such as refractory periods and threshold modulation (see [16]). Such models with time-dependent thresholds and/or resets in some cases can be reduced by an appropriate change of variables to those with constant x_r and x_ϑ (for more details see for example [7, Section 2.5]). Therefore, for simplicity, in this paper we will always assume that $x_\vartheta = 1$ and $x_r = 0$.

Since isolated spikes of a given neuron look alike, often it is assumed that the form of the action potential does not carry any information, but rather it is the structure of the spike train[2] which matters. Therefore, the idea is to study the properties (and, in particular, dynamics) of two special maps associated with the integrate-and-fire models, which carry the information about the distribution of spikings in time, namely, the firing map Φ and its displacement Ψ (cf. Definitions 4.1 and 4.10). It turns out, for example, that the rotation number of a given point with respect to the mapping Φ corresponds to the average interspike interval, whereas its multiplicative inverse describes the firing rate (for more details see Section 6).

Despite the fact that integrate-and-fire models are commonly used, their dynamical behavior has been investigated rigorously in only few papers (see for instance [7,11,16]) and usually under the assumption that the forcing term f (or, in the case of general integrate-and-fire models of the form $\dot{x}(t) = F(t, x(t))$, the function $t \mapsto F(t, x)$) is periodic or uniformly almost periodic (see Definition 2.4 below). Recently, the broader class of stimulating processes than the periodic

[1] In the special case $\sigma = 0$, the LIF model is usually referred to as the *perfect integrator model* (or the PI model for short).

[2] A *spike train* is a chain of action potentials emitted by a single neuron.

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