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Lump solutions to nonlinear partial differential equations via Hirota bilinear forms

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Abstract

Lump solutions are analytical rational function solutions localized in all directions in space. We analyze a class of lump solutions, generated from quadratic functions, to nonlinear partial differential equations. The basis of success is the Hirota bilinear formulation and the primary object is the class of positive multivariate quadratic functions. A complete determination of quadratic functions positive in space and time is given, and positive quadratic functions are characterized as sums of squares of linear functions. Necessary and sufficient conditions for positive quadratic functions to solve Hirota bilinear equations are presented, and such polynomial solutions yield lump solutions to nonlinear partial differential equations under the dependent variable transformations $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$, where x is one spatial variable. Applications are made for a few generalized KP and BKP equations.

MSC: 35Q51; 37K40; 35Q53

Keywords: Soliton; Integrable equation; Hirota bilinear form; Lump solution

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1. Introduction

The Korteweg–de Vries (KdV) equation and the Kadomtsev–Petviashvili (KP) equation are nonlinear integrable differential equations, and their Hirota bilinear forms play a crucial role in generating their soliton solutions, a kind of exponentially localized solutions, describing diverse nonlinear phenomena [9].

By lump functions, we mean analytical rational functions of spatial and temporal variables, which are localized in all directions in space. In recent years, there has been a growing interest in lump function solutions [4,8,10,22], called lump solutions (see, e.g., [1,7,12,25] for typical examples). The KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0 \tag{1.1}$$

admits the following lump solution

$$u = 4\frac{-[x + ay + 3(a^2 - b^2)t]^2 + b^2(y + 6at)^2 + 1/b^2}{\{[x + ay + 3(a^2 - b^2)t]^2 + b^2(y + 6at)^2 + 1/b^2\}^2}, \tag{1.2}$$

where a and $b \neq 0$ are free real constants [21]. Lump functions provide appropriate prototypes to model rogue wave dynamics in both oceanography [23] and nonlinear optics [27]. There are various discussions on general rational function solutions to integrable equations such as the KdV, KP, Boussinesq and Toda equations [2,3,17–19]. It has become a very interesting topic to search for lump solutions or lump-type solutions, rationally localized solutions in almost all directions in space, to nonlinear partial differential equations, through the Hirota bilinear formulation.

In this paper, we would like to characterize positive quadratic functions and analyze positive quadratic function solutions to Hirota bilinear equations. Such polynomial solutions generate lump or lump-type solutions to nonlinear partial differential equations under the dependent variable transformations $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$, where x is one of the spatial variables. We will present sufficient and necessary conditions for positive quadratic functions to solve Hirota bilinear equations, and apply the resulting theory to a few generalized KP and BKP equations.

2. From Hirota bilinear equations to nonlinear equations

Let M be a natural number and $x = (x_1, x_2, \cdots, x_M)^T$ in $\mathbb{R}^M$ be a column vector of independent variables. For $f, g \in C^\infty(\mathbb{R}^M)$, Hirota bilinear derivatives [9] are defined as follows:

$$D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} f \cdot g := \prod_{i=1}^{M} (\partial_{x_i} - \partial_{x_i'})^{n_i} f(x) g(x')|_{x'=x}, \tag{2.1}$$

where $x' = (x_1', x_2', \cdots, x_M')^T$ and $n_i \geq 0$, $1 \leq i \leq M$. For example, we have the first-order and second-order Hirota bilinear derivatives:

$$D_i f \cdot g = f_{x_i} g - f g_{x_i}, \ D_i D_j f \cdot g = f_{x_i,x_j} g + f g_{x_i,x_j} - f_{x_i} g_{x_j} - f_{x_j} g_{x_i}, \tag{2.2}$$

where $1 \leq i, j \leq M$.

One basic property of the Hirota bilinear derivatives is that