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# A limit equation and bifurcation diagrams of semilinear elliptic equations with general supercritical growth ☆

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**Abstract**

We study radial solutions of the semilinear elliptic equation

$$\Delta u + f(u) = 0$$

under rather general growth conditions on $f$. We construct a radial singular solution and study the intersection number between the singular solution and a regular solution. An application to bifurcation problems of elliptic Dirichlet problems is given. To this end, we derive a certain limit equation from the original equation at infinity, using a generalized similarity transformation. Through a generalized Cole–Hopf transformation, all the limit equations can be reduced into two typical cases, i.e., $\Delta u + u^p = 0$ and $\Delta u + e^u = 0$.


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## 1. Introduction and main results

Let $N \geq 3$ and $r := |x|$. In this paper we construct a radial singular solution $u^*(r)$ of the elliptic equation

$$\Delta u + f(u) = 0 \tag{1.1}$$

under rather general growth conditions, and study the intersection number of two radial solutions $\mathcal{Z}_{(0,\infty)}[u(\cdot\,,\rho) - u^*(\cdot)]$. Here, $u(r,\rho)$ is the classical radial solution of (1.1), which satisfies

$$\begin{cases} u'' + \frac{N-1}{r}u' + f(u) = 0, & r > 0, \\ u(0) = \rho, \\ u'(0) = 0, \end{cases} \tag{1.2}$$

and $\mathcal{Z}_I[u_0(\cdot) - u_1(\cdot)]$ denotes the intersection number of the two functions $u_0(r)$ and $u_1(r)$ defined in an interval $I \subset \mathbb{R}$, i.e., $\mathcal{Z}_I[u_0(\cdot) - u_1(\cdot)] = \sharp\{r \in I;\ u_0(r) = u_1(r)\}$. By a radial singular solution $u^*(r)$ of (1.1) we mean that $u^*(r)$ is a classical solution of the equation

$$u'' + \frac{N-1}{r}u' + f(u) = 0 \tag{1.3}$$

on $(0, r_0)$ for some $r_0 > 0$ and $\lim_{r\downarrow 0} u^*(r) = \infty$. We give two applications of the intersection number.

By $F(u)$ we define

$$F(u) := \int_u^\infty \frac{dt}{f(t)}.$$

We assume the following:

$$\text{One of the following (f1-1) or (f1-2) holds:} \tag{f1}$$

$$\text{(a generalization of } u^p\text{)} \quad f(u) \in C^1[0,\infty),\ f(u) > 0 \text{ for } u > 0,\ f(0) = 0,$$
$$f(u) \in C^2(u_0,\infty) \text{ for some } u_0 > 0,\ \lim_{u\downarrow 0} F(u) = \infty, \text{ and } \lim_{u\to\infty} F(u) = 0, \tag{f1-1}$$

$$\text{(a generalization of } e^u\text{)} \quad f(u) \in C^1(\mathbb{R}),\ f(u) > 0 \text{ for } u \in \mathbb{R},$$
$$f(u) \in C^2(u_0,\infty) \text{ for some } u_0 > 0,\ \lim_{u\to -\infty} F(u) = \infty, \text{ and } \lim_{u\to\infty} F(u) = 0. \tag{f1-2}$$

$$\text{There exists the limit } q := \lim_{u\to\infty} \frac{f'(u)^2}{f(u)f''(u)},$$
$$\text{which is denoted by } q \text{ throughout the present paper, and this limit is in } (0,\infty). \tag{f2}$$

Note that the inverse function of $F$, which is denoted by $F^{-1}(u)$, can be defined for $u > 0$, because of (f1). We define the growth rate of $f$ by $p := \lim_{u\to\infty} uf'(u)/f(u)$. By L'Hospital's rule we have