ARTICLE IN PRESS YJDEQ:9045

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Global solvability of the Navier–Stokes equations with a free surface in the maximal L_p-L_q regularity class

Hirokazu Saito

Department of Mathematics, Faculty of Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan

Received 24 July 2017

Abstract

We consider the motion of incompressible viscous fluids bounded above by a free surface and below by a solid surface in the N-dimensional Euclidean space for $N \geq 2$. The aim of this paper is to show the global solvability of the Navier–Stokes equations with a free surface, describing the above-mentioned motion, in the maximal L_p-L_q regularity class. Our approach is based on the maximal L_p-L_q regularity with exponential stability for the linearized equations, and also it is proved that solutions to the original nonlinear problem are exponentially stable.

MSC: primary 35Q30; secondary 76D05

Keywords: Global solvability; Navier–Stokes equations; Free surfaces; Maximal regularity; L_p-L_q framework; Exponential stability

1. Introduction

This paper is concerned with the global solvability of the Navier–Stokes equations with a free surface, describing the motion of incompressible viscous fluids bounded above by a free surface and below by a solid surface in the N-dimensional Euclidean space for $N \geq 2$, in the maximal L_p-L_q regularity class (cf. [36] for the class). Such equations were mathematically treated by Beale [6] for the first time. He proved, in an L_2-in-time and L_2-in-space setting with

E-mail address: hsaito@aoni.waseda.jp.

https://doi.org/10.1016/j.jde.2017.09.045
0022-0396/

Please cite this article in press as: H. Saito, Global solvability of the Navier–Stokes equations with a free surface in the maximal L_p-L_q regularity class, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.09.045

the gravity, the local solvability for large initial data in [6], whereas we prove in the maximal L_p-L_q regularity class the global solvability for small initial data in the case where the gravity is not taken into account in the present paper.

The problem is stated as follows: We are given an initial domain $\Omega \subset \mathbf{R}^N$, occupied by an incompressible viscous fluid, such that

$$\Omega = \{\xi = (\xi', \xi_N) \mid \xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbf{R}^{N-1}, 0 < \xi_N < d\} \quad (d > 0),$$

as well as an initial velocity field $\mathbf{a} = \mathbf{a}(\xi) = (a_1(\xi), \dots, a_N(\xi))^{\mathsf{T}}$[1] of the fluid on Ω. The symbols Γ, S denote the boundaries of Ω such that

$$\Gamma = \{\xi = (\xi', \xi_N) \mid \xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbf{R}^{N-1}, \xi_N = d\},$$
$$S = \{\xi = (\xi', \xi_N) \mid \xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbf{R}^{N-1}, \xi_N = 0\}.$$

We wish to find for each $t \in (0, \infty)$ a transformation $\Theta = \Theta(\cdot, t) : \Omega \ni \xi \mapsto x = \Theta(\xi, t) \in \mathbf{R}^N$, a velocity field $\mathbf{v} = \mathbf{v}(x, t) = (v_1(x, t), \dots, v_N(x, t))^{\mathsf{T}}$ of the fluid, and a pressure field $\pi = \pi(x, t)$ of the fluid so that

$$\partial_t \Theta = \mathbf{v} \circ \Theta, \quad \Theta(\xi, 0) = \xi, \quad \xi \in \Omega, \tag{1.1}$$
$$\Omega(t) = \Theta(\Omega, t), \quad \Gamma(t) = \Theta(\Gamma, t), \quad S = \Theta(S, t), \tag{1.2}$$
$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \operatorname{Div} \mathbf{T}(\mathbf{v}, \pi), \quad x \in \Omega(t), \tag{1.3}$$
$$\operatorname{div} \mathbf{v} = 0, \quad x \in \Omega(t), \tag{1.4}$$
$$\mathbf{T}(\mathbf{v}, \pi)\mathbf{n} = -\pi_0 \mathbf{n}, \quad x \in \Gamma(t), \tag{1.5}$$
$$\mathbf{v} = 0, \quad x \in S, \tag{1.6}$$
$$\mathbf{v}|_{t=0} = \mathbf{a}, \quad \xi \in \Omega, \tag{1.7}$$

where $\mathbf{v} \circ \Theta = (\mathbf{v} \circ \Theta)(\xi, t) = \mathbf{v}(\Theta(\xi, t), t)$.

Here the density of the fluid have been set to 1; $\mathbf{n}$ is the unit outward normal to $\Gamma(t)$; the constant π_0 is the atmospheric pressure, and it is assumed in this paper that $\pi_0 = 0$ without loss of generality. The stress tensor $\mathbf{T}(\mathbf{v}, \pi)$ is given by $\mathbf{T}(\mathbf{v}, \pi) = \mu \mathbf{D}(\mathbf{v}) - \pi \mathbf{I}$, where μ is a positive constant and denotes the viscosity coefficient of the fluid; $\mathbf{I}$ is the $N \times N$ identity matrix; $\mathbf{D}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathsf{T}}$ is the doubled strain tensor. Here and subsequently, we use the following notation for differentiations: Let $f = f(x)$, $\mathbf{g} = (g_1(x), \dots, g_N(x))^{\mathsf{T}}$, and $\mathbf{M} = (M_{ij}(x))$ be a scalar-, a vector-, and an $N \times N$ matrix-valued function on a domain of $\mathbf{R}^N$, respectively, and then for $\partial_j = \partial / \partial x_j$

$$\nabla f = (\partial_1 f, \dots, \partial_N f)^{\mathsf{T}}, \quad \Delta f = \sum_{j=1}^N \partial_j^2 f, \quad \Delta \mathbf{g} = (\Delta g_1, \dots, \Delta g_N)^{\mathsf{T}},$$

[1] $\mathbf{M}^{\mathsf{T}}$ denotes the transposed $\mathbf{M}$.

Please cite this article in press as: H. Saito, Global solvability of the Navier–Stokes equations with a free surface in the maximal L_p-L_q regularity class, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.09.045