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On Birman's sequence of Hardy–Rellich-type inequalities

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Dedicated with great pleasure to Eduard Tsekanovskii on the occasion of his 80th birthday.

Abstract

In 1961, Birman proved a sequence of inequalities $\{I_n\}$, for $n \in \mathbb{N}$, valid for functions in $C_0^n((0,\infty)) \subset L^2((0,\infty))$. In particular, I_1 is the classical (integral) Hardy inequality and I_2 is the well-known Rellich inequality. In this paper, we give a proof of this sequence of inequalities valid on a certain Hilbert space $H_n([0,\infty))$ of functions defined on $[0,\infty)$. Moreover, $f \in H_n([0,\infty))$ implies $f' \in H_{n-1}([0,\infty))$; as a consequence of this inclusion, we see that the classical Hardy inequality implies *each* of the inequalities in Birman's sequence. We also show that for any finite $b > 0$, these inequalities hold on the standard Sobolev space $H_0^n((0,b))$. Furthermore, in all cases, the Birman constants $[(2n-1)!!]^2/2^{2n}$ in these inequalities are sharp and the only function that gives equality in any of these inequalities is the trivial function in $L^2((0,\infty))$ (resp., $L^2((0,b))$). We also show that these Birman constants are related to the norm of a generalized continuous Cesàro averaging operator whose spectral properties we determine in detail.

MSC: primary 26D10, 34A40, 35A23; secondary 34L10

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Contents

1. Introduction

In 1961, M.Š. Birman [8, p. 48], sketched a proof to establish the following sequence of inequalities

$$\int_0^\infty \left|f^{(n)}(x)\right|^2 dx \geqslant \frac{[(2n-1)!!]^2}{2^{2n}} \int_0^\infty \frac{|f(x)|^2}{x^{2n}}\, dx, \quad n \in \mathbb{N}, \tag{1.1}$$

valid for $f \in C_0^n((0,\infty))$, the space of n-times continuously differentiable complex-valued functions having compact support on $(0,\infty)$. Here we employed the well-known symbol, $(2n-1)!! := (2n-1)\cdot(2n-3)\cdots 3\cdot 1$. We denote the inequality in (1.1) by I_n. In particular, I_1 is the classical (integral) Hardy inequality (see [29, Sect. 7.3])

$$\int_0^\infty \left|f'(x)\right|^2 dx \geqslant \frac{1}{4} \int_0^\infty \frac{|f(x)|^2}{x^2}\, dx, \tag{1.2}$$

and I_2 is the Rellich inequality

$$\int_0^\infty \left|f''(x)\right|^2 dx \geqslant \frac{9}{16} \int_0^\infty \frac{|f(x)|^2}{x^4}\, dx. \tag{1.3}$$

We can find no reference in the literature to the general inequality (1.1) prior to the 1966 work of Birman cited above. In [23, pp. 83–84], Glazman gives a detailed proof of (1.1) using the ideas outlined in [8]. In [42, Lemma 2.1], Owen also establishes these inequalities. Each of these authors prove (1.1) for functions on $C_0^n(0,\infty)$. We note in passing that unless $f \equiv 0$, all inequalities (1.1)–(1.3) are strict.