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# Sharp threshold of blow-up and scattering for the fractional Hartree equation

Qing Guo [a], Shihui Zhu [b,c,*]

[a] *College of Science, Minzu University of China, Beijing 100081, China*

[b] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China*

[c] *Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan 610066, China*

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**Abstract**

We consider the fractional Hartree equation in the $L^2$-supercritical case, and find a sharp threshold of the scattering versus blow-up dichotomy for radial data: If $M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < M[Q]^{\frac{s-s_c}{s_c}} E[Q]$ and $M[u_0]^{\frac{s-s_c}{s_c}} \|u_0\|^2_{\dot{H}^s} < M[Q]^{\frac{s-s_c}{s_c}} \|Q\|^2_{\dot{H}^s}$, then the solution $u(t)$ is globally well-posed and scatters; if $M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < M[Q]^{\frac{s-s_c}{s_c}} E[Q]$ and $M[u_0]^{\frac{s-s_c}{s_c}} \|u_0\|^2_{\dot{H}^s} > M[Q]^{\frac{s-s_c}{s_c}} \|Q\|^2_{\dot{H}^s}$, the solution $u(t)$ blows up in finite time. This condition is sharp in the sense that the solitary wave solution $e^{it}Q(x)$ is global but not scattering, which satisfies the equality in the above conditions. Here, $Q$ is the ground-state solution for the fractional Hartree equation.


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*Keywords:* Fractional Schrödinger equation; $L^2$-supercritical; Scattering; Blow-up

* Corresponding author.
*E-mail addresses:* guoqing0117@163.com (Q. Guo), shihuizhumath@163.com (S. Zhu).

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## 1. Introduction

In this paper, we study the fractional Hartree equation, which is the $L^2$-supercritical, nonlinear, fractional Schrödinger equation.

$$iu_t - (-\triangle)^s u + (\frac{1}{|x|^\gamma} * |u|^2)u = 0, \tag{1.1}$$

with $0 < s < 1$ and $2s < \gamma < \min\{N, 4s\}$, where $i$ is the imaginary unit and $u = u(t,x)$: $\mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. The operator $(-\triangle)^s$ is defined by

$$(-\triangle)^s u = \frac{1}{(2\pi)^{\frac{N}{2}}} \int e^{ix\cdot\xi} |\xi|^{2s} \widehat{u}(\xi) d\xi = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}[u](\xi)],$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the Fourier inverse transform in $\mathbb{R}^N$, respectively. The fractional Schrödinger equations were first proposed by Laskin in [28,29] using the theory of functionals over functional measures generated from the Lévy stochastic process and by expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. Here, $s$ is the Lévy index. If $s = \frac{1}{2}$ and $\gamma = 1$, then (1.1) models the dynamics of (pseudo-relativistic) boson stars, where $\frac{1}{|x|}$ is the Newtonian gravitational potential in the appropriate physical units, which is also called the pseudo-relativistic Hartree equation (see [10,30]). The global existence and blow-up have been widely studied in [13,31]. For the classical Hartree equation, a large amount of work has been devoted to the theory of scattering and blow-up, see for example [34–37].

Eq. (1.1) is the $L^2$-supercritical, nonlinear, fractional Schrödinger equation. Indeed, we remark on the scaling invariance of Eq. (1.1). If $u(t,x)$ is a solution of Eq. (1.1), then $u^\lambda(t,x) = \lambda^{\frac{N-\gamma+2s}{2}} u(\lambda^{2s}t, \lambda x)$ is also a solution of Eq. (1.1). This implies that

(1) $\|u^\lambda\|_{L^{p_c}} = \|u\|_{L^{p_c}}$, where $p_c = \frac{2N}{N-\gamma+2s}$. When $\gamma > 2s$, we see that $p_c > 2$, and Eq. (1.1) is called the $L^2$-supercritical, nonlinear, fractional Schrödinger equation.
(2) $\dot{H}^{s_c}$-norm is invariant for Eq. (1.1), i.e., $\|u^\lambda\|_{\dot{H}^{s_c}} = \|u\|_{\dot{H}^{s_c}}$, where $s_c = \frac{\gamma-2s}{2}$.

Now, we impose the initial data,

$$u(0,x) = u_0 \in H^s, \tag{1.2}$$

onto (1.1) and consider the Cauchy problem (1.1)–(1.2). Cho et al. in [7,8] established the local well-posedness in $H^s$ as follows: Let $N \geq 2$, $\frac{1}{2} \leq s < 1$ and $0 < \gamma < \min\{N, 4s\}$. If the initial data $u_0 \in H^s$, then there exists a unique solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) on the maximal time interval $I = [0,T)$ such that $u(t,x) \in C(I; H^s) \bigcap C^1(I; H^{-s})$ and either $T = +\infty$ (global existence) or both $0 < T < +\infty$ and $\lim_{t\to T} \|u(t,x)\|_{H^s} = +\infty$ (blow-up). Moreover, for all $t \in I$, $u(t,x)$ satisfies the following conservation laws.

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