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2D constrained Navier–Stokes equations ☆

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Abstract

We study 2D Navier–Stokes equations with a constraint forcing the conservation of the energy of the solution. We prove the existence and uniqueness of a global solution for the constrained Navier–Stokes equation on $\mathbb{R}^2$ and $\mathbb{T}^2$, by a fixed point argument. We also show that the solution of the constrained equation converges to the solution of the Euler equation as the viscosity ν vanishes.

MSC: primary 35Q30, 37L40; secondary 76M35

Keywords: Navier–Stokes equations; Constrained energy; Periodic boundary conditions; Gradient flow; Euler equations

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1. Introduction

The motivation for this paper is twofold. First, Caglioti et al. in [4] studied the well-posedness and asymptotic behaviour of two dimensional Navier–Stokes equations in the vorticity form with two constraints: constant energy $E(\omega)$ and moment of inertia $I(\omega)$

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega - \nu \operatorname{div}\left[\omega \nabla \left(b\psi + a\frac{|x|^2}{2}\right)\right],$$

which can be rewritten as

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \operatorname{div}\left[\omega \nabla \left(\log \omega - b\psi - a\frac{|x|^2}{2}\right)\right], \tag{1.1}$$

where $\omega = \mathrm{Curl}(u)$, $a = a(\omega)$ and $b = b(\omega)$ are the Lagrange multipliers associated to those constraints and

$$E(\omega) = \int_{\mathbb{R}^2} \psi \omega \, dx, \quad I(\omega) = \int_{\mathbb{R}^2} |x|^2 \omega \, dx, \quad \psi = -\Delta^{-1}\omega.$$

They were able to show the existence of a unique classical global-in-time solution to (1.1) for a family of initial data [4, Theorem 5]. They were also able to prove that the solution to (1.1) converges, as time tends to $+\infty$, to the unique solution of an associated microcanonical variational problem [4, Theorem 8].

Secondly, Rybka [8] and Caffarelli & Lin [3] studied the linear heat equation with constraints. Rybka studied heat flow on a manifold $\mathcal{M}$ given by

$$\mathcal{M} = \left\{ u \in L^2(\Omega) \cap C(\Omega) : \int_{\Omega} u^k(x) \, dx = C_k, \, k = 1, \dots, N \right\},$$

where Ω denotes a connected bounded region in $\mathbb{R}^2$ with smooth boundary. He proved [8, Theorem 2.5] the existence of the unique global solution for the projected heat equation

$$\begin{cases} \frac{du}{dt} = \Delta u - \sum_{k=1}^N \lambda_k u^{k-1} & \text{in } \Omega \subset \mathbb{R}^2, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, & u(0,x) = u_0, \end{cases} \tag{1.2}$$

where $\lambda_k = \lambda_k(u)$ are such that u_t is orthogonal to Span $\{u^{k-1}\}$, for a more regular initial data. He also showed that the solutions to (1.2) converge to a steady state as time tends to $+\infty$.

On the other hand Caffarelli and Lin initially established the existence and uniqueness of a global, energy-conserving solution to the heat equation [3, Theorem 1.1]. They were then able to extend these results to more general family of singularly perturbed systems of nonlocal parabolic equations [3, Theorem 3.1]. Their main result was to prove the strong convergence of the solutions of these perturbed systems to some weak-solutions of the limiting constrained nonlocal heat flows of maps into a singular space.

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