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Self-similar solutions of stationary Navier–Stokes equations

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Abstract

In this paper, we mainly study the existence of self-similar solutions of stationary Navier–Stokes equations for dimension $n = 3, 4$. For $n = 3$, if the external force is axisymmetric, scaling invariant, $C^{1,\alpha}$ continuous away from the origin and small enough on the sphere S^2, we shall prove that there exists a family of axisymmetric self-similar solutions which can be arbitrarily large in the class $C^{3,\alpha}_{loc}(\mathbb{R}^3\backslash 0)$. Moreover, for axisymmetric external forces without swirl, corresponding to this family, the momentum flux of the flow along the symmetry axis can take any real number. However, there are no regular ($U \in C^{3,\alpha}_{loc}(\mathbb{R}^3\backslash 0)$) axisymmetric self-similar solutions provided that the external force is a large multiple of some scaling invariant axisymmetric F which cannot be driven by a potential. In the case of dimension 4, there always exists at least one self-similar solution to the stationary Navier–Stokes equations with any scaling invariant external force in $L^{4/3,\infty}(\mathbb{R}^4)$.

MSC: 35Q30; 76D03

Keywords: Stationary Navier–Stokes equations; Asymptotic behavior; Self-similar solutions

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1. Introduction

The Cauchy problem for the incompressible Navier–Stokes equations in $\mathbb{R}^n$ is given by

$$\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = f \ \text{ in } (0,\infty)\times\mathbb{R}^n \\ \operatorname{div} u = 0 \ \text{ in } (0,\infty)\times\mathbb{R}^n \\ u|_{t=0} = u_0 \ \text{ in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

In the above system, $u(x,t)$ is the velocity field, $p(x,t)$ is the pressure and $f(x,t)$ is the external force. In this paper, we mainly consider the Navier–Stokes equations for $n = 3, 4$. The system (1.1) is invariant under the following scaling

$$\begin{aligned} u(x,t) &\to u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\ p(x,t) &\to p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \\ f(x,t) &\to f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t), \\ u_0(x) &\to u_{0\lambda}(x) = \lambda u_0(\lambda x), \end{aligned}$$

where $\lambda > 0$. In [25], J. Leray proved the global existence of weak solutions $u \in L_t^\infty L_x^2 \bigcap L_t^2 \dot{H}_x^1$ with initial data $u_0 \in L^2(\mathbb{R}^3)$ satisfying $\operatorname{div} u_0 = 0$ in the sense of distributions. However, the problem of regularity and uniqueness of weak solutions is still open. In the same paper, Leray raised the question of the possibility of weak solutions which are singular at a fixed time T and of the following form

$$u(x,t) = \frac{1}{\sqrt{2\alpha(T-t)}} U\left(\frac{x}{\sqrt{2\alpha(T-t)}}\right) \ \ (\alpha > 0,\ t < T)$$

with $U \in L^2(\mathbb{R}^3) \bigcap L^\infty(\mathbb{R}^3)$. The possibility of such self-similar weak solutions $U \in L^3(\mathbb{R}^3)$ was ruled out by Nečas, Ružička and Šverák in [28]; see also Escauriaza–Seregin–Šverák [10]. Tsai [35] excluded the possibility of self-similar weak solutions under more general assumptions; see also D. Chae [6] for Euler equations. The result in [3] implies that suitable self-similar weak solutions are regular if the local scaled energy is sufficiently small. It should be pointed out that the notion of Leray weak solutions can be generalized to more general slow decay solutions which have local finite energy; see Lemarié-Rieusset [23]. There are some previous results on the existence of self-similar solutions of (1.1); see Barraza [1], Cannone and Planchon [4] and Giga and Miyakawa [14]. Recently, Jia and Šverák [19] have shown that if the initial velocity u_0 is homogeneous of degree -1 and of class $C^\alpha_{loc}(\mathbb{R}^3\backslash 0)$ with $\alpha \in (0,1)$, then there exists at least one self-similar suitable weak solution $u \in C^\alpha_{x,t,loc}([0,\infty)\times\mathbb{R}^3\backslash(0,0))$. The existence of discrete forward self-similar solutions was proved by Tsai in [36]. The existence of self-similar solutions with $u_0 \in L^{3,\infty}(\mathbb{R}^3)$ was proved by Bradshaw and Tsai [2]. Korobkov and Tsai [20] showed the existence of self-similar solutions in the upper half space. For other related results, we refer the reader to Xue [37], Chae and Wolf [7].

In this paper, we shall study the existence of self-similar solutions of the stationary Navier–Stokes equations

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