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Loss of boundary conditions for fully nonlinear parabolic equations with superquadratic gradient terms

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Abstract

We study whether the solutions of a fully nonlinear, uniformly parabolic equation with superquadratic growth in the gradient satisfy initial and homogeneous boundary conditions in the classical sense, a problem we refer to as the classical Dirichlet problem. Our main results are: the nonexistence of global-in-time solutions of this problem, depending on a specific largeness condition on the initial data, and the existence of local-in-time solutions for initial data C^1 up to the boundary. Global existence is know when boundary conditions are understood in the viscosity sense, what is known as the generalized Dirichlet problem. Therefore, our result implies loss of boundary conditions in finite time. Specifically, a solution satisfying homogeneous boundary conditions in the viscosity sense eventually becomes strictly positive at some point of the boundary.

MSC: 35D40; 35K55; 35K20; 35P30

Keywords: Loss of boundary conditions; Viscosity solutions; Gradient blow-up; Fully nonlinear parabolic equations; Viscous Hamilton–Jacobi equations; Nonlinear eigenvalues

Contents

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1. Introduction and main results

The present article is a contribution to the study of qualitative properties of viscosity solutions of the so-called Cauchy–Dirichlet problem for the following fully nonlinear parabolic equation with superquadratic growth in the term with gradient dependence:

$$u_t - \mathcal{M}^-(D^2u) = |Du|^p \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x) \quad \text{in } \overline{\Omega}, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying both uniform interior and exterior sphere conditions. While this is not strictly necessary for all our results, it does establish a better connection between our main theorems. See Remarks 3.2 and 6.4. We also assume $p > 2$ throughout, except for certain remarks regarding the case $p \leq 2$ made in this introduction. See also Remark 3.1. Here $\mathcal{M}^-$ denotes one of Pucci's extremal operators, which are defined as follows: let $A, X \in S(n)$, the symmetric $n \times n$ matrices equipped with the usual ordering, I denote the identity matrix, and $0 < \lambda < \Lambda$. Then

$$\mathcal{M}^-(X) = \mathcal{M}^-(X, \lambda, \Lambda) = \inf\{\mathrm{tr}(AX) | \lambda I \leq A \leq \Lambda I\},$$

$$\mathcal{M}^+(X) = \mathcal{M}^+(X, \lambda, \Lambda) = \sup\{\mathrm{tr}(AX) | \lambda I \leq A \leq \Lambda I\}.$$

Alternatively, if we denote by $\lambda_i = \lambda_i(X)$ the eigenvalues of X, then

$$\mathcal{M}^-(X) = \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i,$$

$$\mathcal{M}^+(X) = \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i.$$

Pucci's operators are fundamental to the study of fully nonlinear equations, at once acting as barriers to all equations sharing the same ellipticity constants (owing to the first definition) and allowing fairly explicit computations to be carried out (owing to the second). The Dirichlet condition (1.2) will be considered both in the classical sense and in the generalized sense of viscosity solutions. We will stress the distinction when necessary. Precise definitions and more on this later in this introduction. On the other hand, condition (1.3) is always meant in the classical (pointwise) sense. See Remark 2.5.