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Well-posedness of the free boundary problem in compressible elastodynamics

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Abstract

We study the free boundary problem for the flow of a compressible isentropic inviscid elastic fluid. At the free boundary moving with the velocity of the fluid particles the columns of the deformation gradient are tangent to the boundary and the pressure vanishes outside the flow domain. We prove the local-in-time existence of a unique smooth solution of the free boundary problem provided that among three columns of the deformation gradient there are two which are non-collinear vectors at each point of the initial free boundary. If this non-collinearity condition fails, the local-in-time existence is proved under the classical Rayleigh–Taylor sign condition satisfied at the first moment. By constructing an Hadamard-type ill-posedness example for the frozen coefficients linearized problem we show that the simultaneous failure of the non-collinearity condition and the Rayleigh–Taylor sign condition leads to Rayleigh–Taylor instability.

Keywords: Compressible elastodynamics; Symmetric hyperbolic system; Free boundary problem; Well-posedness; Rayleigh–Taylor instability

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1. Introduction

We consider the equations of elastodynamics governing the motion of a compressible isentropic inviscid elastic fluid [6,8,15]:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p = \operatorname{div}(\rho F F^\top), \\ \dfrac{\mathrm{d}F}{\mathrm{d}t} = \nabla v F, \end{cases} \tag{1}$$

where ρ denotes the density, $v \in \mathbb{R}^3$ the velocity, $F \in \mathbb{M}(3,3)$ the deformation gradient, $\mathrm{d}/\mathrm{d}t = \partial_t + (v \cdot \nabla)$ the material derivative, and the pressure $p = p(\rho)$ is a smooth function of ρ. Moreover, system (1) is supplemented by the identity $\operatorname{div}(\rho F^\top) = 0$ which is the set of the three divergence constraints

$$\operatorname{div}(\rho F_j) = 0 \tag{2}$$

($j = 1,2,3$) on initial data for the Cauchy problem, where $F_j = (F_{1j}, F_{2j}, F_{3j})$ is the vector field corresponding to the jth column of the deformation gradient, i.e., one can show that if the initial data for (1) satisfy (2), then the divergence constraints (2) hold for all $t > 0$. The first-order system (1) written in the Eulerian coordinates describes the motion of elastic waves in a compressible material for which the Cauchy stress tensor has the form $\rho F F^\top$ corresponding to the elastic energy $W(F) = \frac{1}{2}|F|^2$ for the Hookean linear elasticity. At last, we note that system (1) arises as the inviscid limit of the equations of compressible viscoelasticity [6,8,15] of Oldroyd type [22,23] (see, e.g., [11–14,24,25,27] and references therein for various aspects of analysis of these equations).

Taking into account the divergence constraints (2), we easily symmetrize system (1) by rewriting it as

$$\begin{cases} \dfrac{1}{\rho c^2}\dfrac{\mathrm{d}p}{\mathrm{d}t} + \operatorname{div} v = 0, \\ \rho\dfrac{\mathrm{d}v}{\mathrm{d}t} + \nabla p - \rho \displaystyle\sum_{j=1}^{3}(F_j \cdot \nabla)F_j = 0, \\ \rho\dfrac{\mathrm{d}F_j}{\mathrm{d}t} - \rho(F_j \cdot \nabla)v = 0, \end{cases} \tag{3}$$

where $c^2 = p'(\rho)$ is the square of the sound speed. Equations (3) form the symmetric system

$$A_0(U)\partial_t U + \sum_{k=1}^{3} A_k(U)\partial_k U = 0 \tag{4}$$

for $U = (p, v, F_1, F_2, F_3)$, with $A_0 = \operatorname{diag}(1/(\rho c^2), \rho I_{12})$ and

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