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Regularity for the evolution of p-harmonic maps

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Abstract

This paper presents our study of regularity for p-harmonic map heat flows. We devise a monotonicity-type formula of scaled energy and establish a criterion for a uniform regularity estimate for regular p-harmonic map heat flows. As application we show the small data global in the time existence of regular p-harmonic map heat flow.

MSC: primary 35D10; secondary 35D05, 35B65, 35K50, 35K65

Keywords: p-Laplacian system; p-Harmonic map heat flow; p-Harmonic map; Regularity; Monotonicity estimate

1. Introduction

Let $\Omega \subset \mathbb{R}^m$ $(m \geq 2)$ be a bounded domain with a smooth boundary, and let $\mathcal{N}$ be an n-dimensional smooth compact Riemannian manifold without a boundary assumed to be isometrically embedded in $\mathbb{R}^l$ $(l > n)$. For a smooth map u from Ω to $\mathcal{N} \subset \mathbb{R}^l$ we consider the p-energy functional

$$E_p(u) = \int_\Omega \frac{1}{p} |Du|^p \, dx, \tag{1.1}$$

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where $p \geq 2$, $u = \left(u^i\right)$, $i = 1, \dots, l$, is a vector-valued function, defined on Ω with values in $\mathcal{N} \subset \mathbb{R}^l$. $D_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, \dots, m$, $Du = \left(D_\alpha u^i\right)$ is the gradient of a map u, $|Du|^2 = \sum_{\alpha=1}^m \sum_{i=1}^l (D_\alpha u^i)^2$. A p-harmonic map is defined to be a critical point, which is a solution of the Euler–Lagrange equation of the p-energy

$$-\mathrm{div}\left(|Du|^{p-2} Du\right) = |Du|^{p-2} A(u)(Du, Du), \tag{1.2}$$

where $A(\cdot)(\cdot, \cdot) = \left(\sum_{j,\,k=1}^l A^i_{j\,k}(u) Du^j \cdot Du^k\right)$ is the second fundamental form of $\mathcal{N} \subset \mathbb{R}^l$ (if necessary, the manifold $\mathcal{N}$ is assumed to be orientable). We look for critical points of the p-energy (1.1), the p-harmonic maps, by considering the negative gradient flow of the p-energy referred to as the p-harmonic map heat flow

$$\partial_t u - \mathrm{div}\left(|Du|^{p-2} Du\right) = |Du|^{p-2} A(u)(Du, Du). \tag{1.3}$$

In this paper we consider the regularity of solutions of the p-harmonic map heat flow (1.3).

Here we derive the Euler-Lagrange equation of (1.1) and the gradient flows. Let u be a smooth map from Ω to $\mathcal{N}$. Let ϕ be a smooth $\mathbb{R}^l$-vector valued function on Ω with compact support. Take a comparison map $\Pi(u + \tau\phi)$, $\tau \in \mathbb{R}$. Here Π: $\mathbb{R}^l \supset \mathcal{O}(\mathcal{N}) \to \mathcal{N} \subset \mathbb{R}^l$ is the nearest point projection from a tubular neighborhood $\mathcal{O}(\mathcal{N}) \subset \mathbb{R}^l$ of $\mathcal{N}$, to $\mathcal{N}$. For any sufficient small number τ, $|\tau| \ll |\phi|_\infty$, the map $u + \tau\phi$ has its value in $\mathcal{O}(\mathcal{N})$ and thus, $\Pi(u + \tau\phi) \in \mathcal{N}$ is an admissible comparison map. We compute the first variation (the Gâteaux derivative) to have

$$\frac{d}{d\tau} E\left(\Pi(u + \tau\phi)\right)\Big|_{\tau=0} = \int_\Omega |Du|^{p-2} \left(Du \cdot D\phi + \phi \cdot \frac{d^2\Pi}{du^2}(u)(Du, Du)\right) dx. \tag{1.4}$$

Integration by parts in the formula (1.4) $= 0$ gives us the Euler–Lagrange equation (1.2). Here,

$$\frac{d^2\Pi}{du^2}(u)(Du, Du) = -\left(A^i(u)(Du, Du)\right) = \left(\sum_{j,\,k=1}^l \frac{d^2\Pi^i}{du^j\, du^k}(u) Du^j \cdot Du^k\right), \quad i = 1, 2, \dots, l, \tag{1.5}$$

is the second fundamental form of $\mathcal{N} \subset \mathbb{R}^l$. For the p-energy for smooth maps $u \in C^\infty(\Omega, \mathcal{N})$,[2] its gradient-like vector field $\nabla E(u)$ can be written as[3]

$$\langle \nabla E(u), \phi \rangle = \frac{d}{d\tau} E\left(\Pi(u + \tau\phi)\right)\Big|_{\tau=0}$$

and then, by (1.4)

[2] $C^\infty(\Omega, \mathcal{N})$ is a Banach manifold.

[3] $\langle \nabla E(u), \cdot \rangle$ is a bounded linear functional on a tangent space $\bigcup_{u \in \mathcal{X}} C^\infty(\Omega, T_u(\mathcal{N}))$ of the Banach manifold $\mathcal{X} := C^\infty(\Omega, \mathcal{N})$.

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