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Asymptotic behavior of equilibrium states of reaction–diffusion systems with mass conservation

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Abstract

We deal with a stationary problem of a reaction–diffusion system with a conservation law under the Neumann boundary condition. It is shown that the stationary problem turns to be the Euler–Lagrange equation of an energy functional with a mass constraint. When the domain is the finite interval $(0, 1)$, we investigate the asymptotic profile of a strictly monotone minimizer of the energy as d, the ratio of the diffusion coefficient of the system, tends to zero. In view of a logarithmic function in the leading term of the potential, we get to a scaling parameter κ satisfying the relation $\varepsilon := \sqrt{d} = \sqrt{\log \kappa}/\kappa^2$. The main result shows that a sequence of minimizers converges to a Dirac mass multiplied by the total mass and that by a scaling with κ the asymptotic profile exhibits a parabola in the nonvanishing region. We also prove the existence of an unstable monotone solution when the mass is small.

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1. Introduction

In the fields of population biology and cell biology concentration phenomena are often observed by aggregation of species and chemical substances respectively. One of the well known models is a Keller–Segel chemotaxis model [21] in which spiky patterns appears by the aggregation of cellular slime mold, though it blows up in a higher dimensional domain (for instance, see [17], [5], [23], [20], [25] and the references therein). In this model the total mass of the slime mold is conserved in a reasonable setting. On the other hand in a study for the cell polarity the authors [19] and [7] proposed simple conceptual models to describe the concentration phenomenon induced by a different mechanism from the chemotaxis model, though the mass conservation property shares in the both models. After their contribution, mathematical studies for the conceptual models are developed in [16], [15], [8], [10] and [9] (see also [13], [14], [11] and [12]). In particular, it is shown in [16], [15] and [8] that the spiky pattern is certainly stable in their model equations.

Motivated by those studies, we are concerned with the following reaction–diffusion system:

$$\begin{cases} u_t = d\Delta u - g(u+\gamma v) + v, \\ v_t = \Delta v + g(u+\gamma v) - v, \end{cases} \qquad x \in \Omega, \tag{1.1}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $0 < d < 1$. We note that the diffusion coefficients of u and v equations are normalized: 1 in the v-equation and d in the u-equation where d stands for the ratio of the two diffusion coefficients. For specific cases $g(u) = au/(u^2 + b)$ $(\gamma = 0)$ and $g(w) = w/(w+1)^2$ $(\gamma = 1)$ are provided by [19], where a, b are positive constants.

Here, we deal with the case for $\gamma = 1$ and fix the function $g(w)$ as

$$g(w) = \frac{w}{(w+1)^2}.$$

It is known that there exists a unique nonnegative classical solution satisfying the initial condition

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad u_0, v_0 \in C^0(\overline{\Omega}), \ \ u_0(x) \geq 0, v_0(x) \geq 0 \ (x \in \overline{\Omega})$$

(see [8] and [9]). Under the evolution of the system, the total mass is conserved:

$$\int_\Omega (u(x,t) + v(x,t))dx = \int_\Omega (u_0(x) + v_0(x_0))dx \qquad (t \geq 0).$$

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