ARTICLE IN PRESS YJDEQ:9016

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Approximate controllability results for viscoelastic flows with infinitely many relaxation modes

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Received 6 March 2017

Abstract

We prove results on approximate controllability for linear viscoelastic flows, with a localized distributed control in the momentum balance equation. The constitutive law is a multimode Maxwell or Jeffreys model with an infinite number of relaxation modes.

Keywords: Linear viscoelasticity; Approximate controllability; Laplace transform

1. Introduction

We consider the linearized equations of incompressible viscoelastic flow in a domain $\Omega \subset \mathbb{R}^d$. With velocity u, density ρ, pressure p and the extra stress tensor τ, the balance equations of mass and momentum are

$$\begin{aligned} \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0, T), \\ \rho u_t &= \nabla \cdot \tau - \nabla p \quad \text{in } \Omega \times (0, T). \end{aligned} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jde.2017.09.016
0022-0396/

To complete the system, we need a constitutive law relating the stress tensor to the motion. For linear viscoelasticity, it has the form

$$\tau(x,t) = \int_0^\infty G(s)(\nabla u(x,t-s) + (\nabla u(x,t-s))^T)\, ds. \tag{1.2}$$

The function G is called the stress relaxation modulus. If we formally set $G(s) = \delta(s)$, we recover the constitutive law for a Newtonian fluid. In general, the quantity $G(0)$ is called the instantaneous stress modulus, and the integral of G is the viscosity.

It is usually assumed in rheology that G is completely monotone. Maxwell's theory of linear elasticity assumes that $G(s) = \kappa \exp(-\lambda s)$; this allows us to replace the integral constitutive law by the ordinary differential equation

$$\tau_t + \lambda\tau = 2\kappa Du, \quad Du = \frac{1}{2}(\nabla u + (\nabla u)^T), \tag{1.3}$$

where $1/\lambda$ is the relaxation time and κ is the instantaneous stress modulus. Jeffreys model assumes the stress to be a linear combination of a Maxwell stress term and a Newtonian stress term

$$G(s) = \eta\delta(s) + \kappa e^{-\lambda s},$$

where $\eta > 0$ is the Newtonian contribution to the viscosity. In order to fit data on real fluids, it is usually necessary to include several relaxation modes. Thus natural generalizations of Maxwell and Jeffreys models assume a linear combination of several or infinitely many contributions to the stress, which are each governed by an equation of the form (1.3).

In this paper, we specialize G to be a discrete sum of exponentials. The goal of this paper is to establish approximate controllability for Maxwell or Jeffreys models with infinitely many relaxation modes. That is, we add a control to the momentum equation, which is localized in a subdomain $\mathcal{O}$, i.e. we have the equations

$$\begin{aligned} \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0,T), \\ \rho u_t &= \nabla \cdot \tau - \nabla p + f\chi_{\mathcal{O}} \quad \text{in } \Omega \times (0,T). \end{aligned} \tag{1.4}$$

We want to establish that, for given initial data, we can reach a final state at $t = T$ which is as close to a given state as we wish.

The initial approach to control of viscoelastic media was by perturbation of the elastic case and control of the displacement (in the case of solids) and velocity, see e.g. [7–11]. However, the future evolution of a viscoelastic material depends not only on displacement and velocity but also on the residual stresses. That is why in [5], controllability results of linear Maxwell and Jeffreys fluids were announced which included control of the stress as well as the motion. No complete proofs were given in [5]. In [12], one-dimensional shear flows of multimode linear Maxwell and Jeffreys fluids are considered, with a distributed control localized on a subinterval. Exact controllability for single-mode Maxwell fluids and approximate controllability of multimode Maxwell and Jeffreys fluids are established. For Jeffreys or multimode Maxwell models, approximate controllability is the best one can hope for, since observability estimates in any reasonable Sobolev norms cannot hold (see Theorem 5 in [15]).

Please cite this article in press as: D. Mitra et al., Approximate controllability results for viscoelastic flows with infinitely many relaxation modes, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.09.016