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The linearization of periodic Hamiltonian systems with one degree of freedom under the Diophantine condition

Nina Xue [a,b], Xiong Li [a,*,1]

[a] School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China
[b] School of Mathematics and Information Sciences, Weifang University, Weifang, Shandong, 261061, PR China

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Abstract

In this paper we are concerned with the periodic Hamiltonian system with one degree of freedom, where the origin is a trivial solution. We assume that the corresponding linearized system at the origin is elliptic, and the characteristic exponents of the linearized system are $\pm i\omega$ with ω be a Diophantine number, moreover if the system is formally linearizable, then it is analytically linearizable. As a result, the origin is always stable in the sense of Liapunov in this case.

MSC: 37H10; 34F05; 34K30; 34D20

Keywords: Periodic Hamiltonian systems; Normal form; Liapunov stability

1. Introduction

We consider a Hamiltonian system with one degree of freedom

$$\dot{x} = \frac{\partial H}{\partial y}(x, y, t), \quad \dot{y} = -\frac{\partial H}{\partial x}(x, y, t), \tag{1.1}$$

* Corresponding author.
E-mail address: xli@bnu.edu.cn (X. Li).
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where $H(0,0,t) = H_x(0,0,t) = H_y(0,0,t) = 0$, the dot indicates differentiation with respect to the time t. The Hamiltonian function H is continuous and 2π-periodic in t, real analytic in a neighborhood of the origin $(x, y) = (0, 0)$, and the Taylor series of H in a neighborhood of the origin is assumed to be

$$H(x, y, t) = H_2 + H_3 + \cdots + H_j + \cdots, \tag{1.2}$$

where

$$H_j = \sum_{\mu+\nu=j} h_{\mu\nu}(t)x^{\mu}y^{\nu}, \quad j = 2, 3, \cdots,$$

and the coefficients $h_{\mu\nu}(t)$ are 2π-periodic with respect to the time t.

Initially we want to investigate the problem about the stability of the trivial solution of the periodic Hamiltonian system (1.1). There are plenty of works about the stability of the trivial solution, and we can refer to [10], [23] for a detailed description. For recent developments, one may consult [5], [15] and the references therein. For time-periodic Lagrangian equations, an analytical method called the third order approximation, has been developed recently by Ortega in a series of papers [12], [13], [14], [16]. After that, some researcher have extend the applications of the third approximation, and some stability results for several types of Lagrangian equations have been established. We refer the reader to [4], [15] for the forced pendulum equation, and [4], [24], [25] for some singular equations.

It is well known that for periodic Hamiltonian system (1.1), if there exists a characteristic exponent of the linearized system with nonzero real part, then the trivial solution is unstable, thus we assume that the linearized system is stable and that the characteristic exponents are pure imaginary, say $\pm i\omega$.

In the general elliptic case, Arnold [1] and Moser [9], [23] proved that the solution is stable if certain nondegeneracy conditions are fulfilled. In the case of resonance, i.e., ω is a rational number, the above result cannot be applied directly. For this case, there also are many results, see [2], [3], [6], [7], [20], [21], [22] and the references therein. For example, Mansilla [6] obtained some sufficient conditions for stability and instability of the trivial solution by using Moser's twist theorem and Liapunov theorem, respectively. Bardin [2] studied the degenerate case by using Lie normal form, and established some general criteria to solve the stability problem. The basic method of these results is that by considering up to terms of a certain order in the normal form, then the sufficient conditions for stability and instability are obtained, respectively.

On the other hand, Rüssmann [11] considered a real analytic area-preserving mapping near the fixed point $(0, 0)$ of the form

$$\begin{cases} x_1 = f(x, y) = x\cos\gamma_0 - y\sin\gamma_0 + \cdots, \\ y_1 = f(x, y) = x\sin\gamma_0 + y\cos\gamma_0 + \cdots, \end{cases} \tag{1.3}$$

where f and g are convergent power series in x, y with real coefficients and $\frac{\gamma_0}{2\pi}$ is an irrational real number. He proved that if (1.3) is formally linearizable, then it is analytically linearizable under the Bruno condition. We recall the main ideas of the proof. Rüssmann first studies the process of formal normalization using a functional iterative approach, then he constructs a formal iteration process converging to a zero of the operator $\mathcal{F} := f \circ \varphi - \varphi \circ \Lambda$ (where Λ is the linear