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Travelling waves in dilatant non-Newtonian thin films

Joachim Escher, Christina Lienstromberg *

Institute for Applied Mathematics, Leibniz Universität Hannover, Germany

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Abstract

We prove the existence of a travelling wave solution for a gravity-driven thin film of a viscous and incompressible dilatant fluid coated with an insoluble surfactant. The governing system of second order partial differential equations for the film's height h and the surfactant's concentration γ are derived by means of lubrication theory applied to the non-Newtonian Navier–Stokes equations.

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1. Introduction

Non-Newtonian fluids are ubiquitous in Nature and technical applications. In contrast to Newtonian fluids non-Newtonian fluids exhibit a shear rate dependent viscosity. A commonly used classification for non-Newtonian viscosities is to distinguish between shear-thinning, shear-thickening and generalised Newtonian fluids. While shear-thinning fluids – whose viscosity decreases with increasing shear stress – appear as natural as well as industrial liquids, shear-thickening fluids are mostly observed as suspensions. In this work we treat only shear-thickening or *dilatant* fluids, i.e. those whose viscosity increases with increasing shear stress. We consider a thin film of a viscous, incompressible dilatant fluid on a horizontal impermeable bottom, carrying

* Corresponding author.
E-mail addresses: escher@ifam.uni-hannover.de (J. Escher), lienstromberg@ifam.uni-hannover.de (C. Lienstromberg).

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a layer of insoluble *surface active agents*, also named *surfactant* on its surface. As a consequence of the *Marangoni effect*, saying that the surfactant spread from places with lower surface tension to places with higher surface tension, the presence oft surfactant leads to a diminution of the surface tension. In mathematical parlance, the surface tension is a decreasing function of the surfactant's concentration. Starting from the Navier–Stokes equation we use lubrication theory in oder to describe the hydrodynamic behaviour of the thin film. More precisely, we denote by $\gamma(t,x)$ and $h(t,x)$ the surfactant's concentration and the film's height at time $t \geq 0$ and position $x \in \mathbb{R}$, respectively, by G the gravitational force, and by d_0 a positive diffusion coefficient. Furthermore, we specify the viscosity function to be a first-order approximation

$$\mu(d) = \mu_0 + \mu_1 d, \quad d \geq 0,$$

of a general dilatant non-Newtonian fluid, where μ_0 and μ_1 are positive.[1] Finally, introducing the surface tension $\sigma = \sigma(\gamma(t,x))$ we may formulate the evolution equations

$$h_t + \frac{d}{dx}\left(\frac{1}{2\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x h^2 - \frac{G}{3\mu_0} h_x h^3\right) = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{1}$$

$$\gamma_t + \frac{d}{dx}\left(\frac{1}{\mu_0} h\gamma\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{2\mu_0} h^2 h_x \gamma - d_0\gamma_x\right) = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{2}$$

for the film's height h and the surfactant's concentration γ. Note that we deal with a gravity driven system of second order, neglecting fourth-order capillary effects.

To the best of our knowledge there are no rigorous analytic results available about non-Newtonian thin-film equations including the presence of surfactant. For results on non-Newtonian thin films without the influence of surfactant we refer the reader to the works [12, 13], where the spreading of general power-law fluids and travelling wave solutions for thin films of power-law fluids are considered, respectively. In [2] the authors study self-similar solutions in free-boundary problems for non-Newtonian power-law fluids. Moreover, in [15] the authors use lubrication approximation to derive the governing equations of a power law non-Newtonian liquid flowing on an inclined plane and they study qualitative properties of the corresponding travelling waves solutions.

For the Newtonian case, we mention the paper [8], where the authors study a coupled system of parabolic equations, consisting of a degenerate fourth-order equation for the film's height and a second-order equation for the surfactant's concentration. For further results on the Newtonian case see [3,6,9,11,14,16] and the references therein. Finally, we refer to the contribution [7], in which the authors prove the existence of travelling waves in a Newtonian thin film with surfactant. In our work we verify that the results of the latter case are even true in the non-Newtonian case of a shear-thickening fluid. More precisely, we prove the following result:

Theorem. *Given any $c > 0$, there exists a C^2-travelling wave solution to (1)–(2) connecting a fully coated state, ($\gamma \sim 1$) with an uncoated state ($\gamma \sim 0$). More precisely, there is a pair of functions $(h_0, \gamma_0) \in C^2(\mathbb{R}, \mathbb{R}^2)$ such that*

$$\lim_{\xi \to -\infty} \gamma_0(\xi) = 1, \quad \lim_{\xi \to \infty} \gamma_0(\xi) = 0$$

[1] A further smallness condition on the ratio μ_1/μ_0 is need in our analysis.