ARTICLE IN PRESS YJDEQ:9025

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Space-time asymptotics of the two dimensional Navier–Stokes flow in the whole plane

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Received 27 April 2017; revised 12 August 2017

Abstract

We consider the space-time behavior of the two dimensional Navier–Stokes flow. Introducing some qualitative structure of initial data, we succeed to derive the first order asymptotic expansion of the Navier–Stokes flow without moment condition on initial data in $L^1(\mathbb{R}^2) \cap L^2_\sigma(\mathbb{R}^2)$. Moreover, we characterize the necessary and sufficient condition for the rapid energy decay $\|u(t)\|_2 = o(t^{-1})$ as $t \to \infty$ motivated by Miyakawa–Schonbek [21]. By weighted estimated in Hardy spaces, we discuss the possibility of the second order asymptotic expansion of the Navier–Stokes flow assuming the first order moment condition on initial data. Moreover, observing that the Navier–Stokes flow $u(t)$ lies in the Hardy space $H^1(\mathbb{R}^2)$ for $t > 0$, we consider the asymptotic expansions in terms of Hardy-norm. Finally we consider the rapid time decay $\|u(t)\|_2 = o(t^{-\frac{3}{2}})$ as $t \to \infty$ with cyclic symmetry introduced by Brandolese [2].

MSC: 35Q30; 76D05

Keywords: Time decay; Asymptotic expansion; Large time behavior

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http://dx.doi.org/10.1016/j.jde.2017.09.025
0022-0396/

Please cite this article in press as: T. Okabe, Space-time asymptotics of the two dimensional Navier–Stokes flow in the whole plane, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.09.025

1. Introduction

We consider the Navier–Stokes equations in $\mathbb{R}^2$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^2, \end{cases} \tag{N-S}$$

where $u = u(x,t) = \big(u_1(x,t), u_2(x,t)\big)$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, respectively, while $u_0 = u_0(x) = \big(u_0^1(x), u_0^2(x)\big)$ denotes the given initial velocity.

In two dimensional case, it is well-known that the (unique) weak solution u in the class $L^\infty\big(0,\infty\,;\, L^2(\mathbb{R}^2)\big)$ of (N-S) is actually the strong solution and satisfies the following integral equation:

$$u(t) = e^{t\Delta} u_0 - \int_0^t P\nabla \cdot e^{(t-s)\Delta}(u \otimes u)(s)\, ds, \qquad t > 0, \tag{IE}$$

where $\{e^{t\Delta}\}_{t \ge 0}$ is the heat semigroup, $P = (P_{jk})_{j,k=1}^2$ is the Helmholtz or the Fujita–Kato bounded projection onto the solenoidal vector fields and $u \otimes u = (u_j u_k)_{j,k=1}^2$.

Since Leray's celebrated paper [16], the decay problem has been one of main interests in mathematical fluid mechanics. Especially, the algebraic decay with respect to the time variable is investigated by, for instance, Schonbek [23–26], Kajikiya and Miyakawa [13], Wiegner [30,31]. Indeed for the general dimension $n \ge 2$, under the moment condition on u_0, i.e.,

$$\int_{\mathbb{R}^n} (1 + |x|)|u_0(x)|\, dx < \infty, \tag{1.1}$$

the upper bound:

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n+2}{4}}, \qquad t \ge 0 \tag{1.2}$$

is observed for some weak solutions of the Navier–Stokes equations. See also [5–7,10,17,18,2]. Here the rate $(1+t)^{-\frac{n+2}{4}}$ of the energy decay is known as the critical rate of the nonlinear term compared with the Stokes flow. For instance, under the moment condition (1.1), Carpio [3], Fujigaki and Miyakawa [8], Miyakawa and Schonbek [21] considered the asymptotic expansion in terms of the heat kernel function $(4\pi t)^{-n/2} \exp(-|x|^2/4t)$, where the leading terms were definitely described. As for the characterization of the leading order terms of the Stokes flow, the moment condition (1.1) seems to be necessary and essential not only to derive the rapid decay (1.2) but also to obtain the first order asymptotic expansion of the Navier–Stokes flow.

In this paper, our aim is to obtain the space-time asymptotics and to characterize the leading order terms using the heat kernel function without any moment condition on initial data like (1.1). Alternatively, we introduce the following profile of initial data:

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