ARTICLE IN PRESS
YJDEQ:9061

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Forced waves of the Fisher–KPP equation in a shifting environment ☆

Henri Berestycki [a], Jian Fang [b,*]

[a] *École des hautes études en sciences sociales, PSL Research University, Centre d'analyse et de mathématique sociales, CNRS, Paris, France*

[b] *Institute for Advanced Study in Mathematics and Department of Mathematics, Harbin Institute of Technology, Harbin, China*

Received 13 March 2017; revised 16 October 2017

Abstract

This paper concerns the equation

$$u_t = u_{xx} + f(x - ct, u), \quad x \in \mathbb{R}, \tag{0.1}$$

where $c \geq 0$ is a forcing speed and $f : (s, u) \in \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is asymptotically of KPP type as $s \to -\infty$. We are interested in the questions of whether such a forced moving KPP nonlinearity from behind can give rise to traveling waves with the same speed and how they attract solutions of initial value problems when they exist. Under a sublinearity condition on $f(s, u)$, we obtain the complete existence and multiplicity of forced traveling waves as well as their attractivity except for some critical cases. In these cases, we provide examples to show that there is no definite answer unless one imposes further conditions depending on the heterogeneity of f in $s \in \mathbb{R}$.

MSC: 35C07; 35B40; 35K57; 92D25

☆ The research leading to these results was supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement No. 321186 – ReaDi –Reaction–Diffusion Equations, Propagation and Modelling held by Henri Berestycki. This work was also partially supported by the NSF of Heilongjiang province (LC2017002) and the French National Research Agency (ANR), within the project NONLOCAL ANR-14-CE25-0013. We thank the anonymous referee for the comments that improve the paper.

* Corresponding author.
E-mail addresses: hb@ehess.fr (H. Berestycki), jfang@hit.edu.cn (J. Fang).

https://doi.org/10.1016/j.jde.2017.10.016

Please cite this article in press as: H. Berestycki, J. Fang, Forced waves of the Fisher–KPP equation in a shifting environment, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.016

Keywords: Shifting environment; Fisher–KPP equation; Traveling waves; Long time behavior

1. Introduction

This paper deals with the equation

$$u_t = u_{xx} + f(x - ct, u), \quad x \in \mathbb{R}, \tag{1.1}$$

where $c \geq 0$ and $f \in C^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ is assumed to have the following properties:

$$f(s, 0) = 0 \text{ for all } s \in \mathbb{R}; \tag{1.2}$$

$$\text{the limits } f(\pm\infty, u) \text{ and } \partial_u f(\pm\infty, u) \text{ exist and are continuous for } u \geq 0; \tag{1.3}$$

$$f(-\infty, u) = 0 \text{ has a unique positive solution } \alpha; \tag{1.4}$$

$$f(s, u)/u \text{ is non-increasing in } u > 0 \text{ for any } s \in \mathbb{R}; \tag{1.5}$$

$$\text{there exists } M > 0 \text{ such that } f(s, u) < 0 \text{ for } u \geq M, \text{ for all } s \in \mathbb{R}. \tag{1.6}$$

A typical example of such a nonlinearity is $f(s, u) = u(a(s) - u)$, where a is a smooth function and has limits at $\pm\infty$ with $a(-\infty) > 0$. Here $a(s)$ may have negative limit at $+\infty$ and may also change sign when s is away from $\pm\infty$. Another example is $f(s, u) = b(s)u(1 - u)$, where $b > 0$ has positive limits at $\pm\infty$.

A forced wave solution of (1.1) has the form $u(t, x) = U_c(x - ct)$, where c is the forced speed and U_c is the profile satisfying

$$U_c''(x) + cU_c'(x) + f(x, U_c(x)) = 0, \quad x \in \mathbb{R}. \tag{1.7}$$

The main purpose of this paper is to study under what conditions a forward shifting KPP nonlinearity gives rise to this kind of forced wave solutions. Let S be the set of all positive and bounded solutions of (1.7). Our goal here is to draw a complete picture of S and to study the attractivity of forced waves for the initial value problem of (1.1). Let us note that in our proofs to establish the main results about (1.7) we employ both ODE and PDE arguments, even though we treat an ODE.

We first recall some related developments and unsolved questions about this problem.

If $f(s, u)$ does not depend on $s \in \mathbb{R}$, that is, $f(s, u) \equiv g(u)$ is *homogeneous*, then under the assumptions (1.2)–(1.6) the nonlinearity $g(u)$ is of KPP type, that is,

$$g \text{ has a unique positive zero } \alpha \text{ and } g(u) \leq g'(0)u \text{ for } u \geq 0. \tag{1.8}$$

This equation $u_t = u_{xx} + g(u)$ in such a case has been extensively studied, since the classical works of Fisher [15] and Kolmogorov, Petrovsky and Piskunov (KPP) [23]. It is well-known that $c^* := 2\sqrt{g'(0)}$ is the minimal speed for traveling waves solution $U_c(x - ct)$, which satisfies $U_c'' + cU_c' + g(U_c) = 0$ with $U_c(+\infty) = 0$ and $U_c(-\infty) = \alpha$. Such a U_c is unique up to translations. Further, $\lim_{x\to+\infty} U_c(x)x^{1-m_c}e^{-\lambda_c x}$ is a positive number, where λ_c is the largest negative solution of $\lambda^2 + c\lambda + g'(0) = 0$ and m_c is its multiplicity. Aronson and Weinberger [2] showed that c^* is also the spreading speed of solutions of (1.1) having compactly supported initial data.

Please cite this article in press as: H. Berestycki, J. Fang, Forced waves of the Fisher–KPP equation in a shifting environment, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.016