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Stability and instability of stationary solutions for sublinear parabolic equations ☆

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Abstract

In the present paper, we study the initial boundary value problem of the sublinear parabolic equation. We prove the existence of solutions and investigate the stability and instability of stationary solutions. We show that a unique positive and a unique negative stationary solutions are exponentially stable and give the exact exponent. We prove that small stationary solutions are unstable. For one space dimensional autonomous equations, we elucidate the structure of stationary solutions and study the stability of all stationary solutions.

MSC: 35K20; 35K58

Keywords: Sublinear parabolic equation; Semilinear parabolic equation; Stability; Instability; Stationary solutions; Lyapunov functional

1. Introduction

We study the stability and instability of stationary solutions for the sublinear parabolic equation

$$\begin{aligned} u_t - \Delta u = f(x, u) \quad &\text{in } \Omega \times (0, \infty), \\ u = 0 \quad &\text{on } \partial\Omega \times (0, \infty), \end{aligned} \tag{1.1}$$

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$$u(x,0) = u_0(x) \qquad \text{in } \Omega,$$

where $u_t = \partial u/\partial t$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Typical examples of $f(x,u)$ in (1.1) are the following:

$$a(x)|u|^{p-1}u, \quad -a(x)u\log|u|, \quad a(x)|u|^{p-1}ue^{-|u|}, \quad a(x)|u|^{p-1}u - b(x)|u|^{q-1}u,$$

where $0 < p < 1 < q$ and $a(x)$ and $b(x)$ are positive continuous functions on $\overline{\Omega}$. The above functions $f(x,u)$ satisfy that $f(x,u)/u$ is decreasing with respect to u in $(0,\infty)$ and increasing in $(-\infty,0)$. When f satisfies this condition, we call $f(x,u)$ *sublinear*. In the paper [4], we have studied the stability of stationary solutions for the case $f(x,u) = |u|^{p-1}u$ with $0 < p < 1$. For this nonlinearity, we note that the uniqueness of solutions does not hold for the problem (1.1) with the initial data $u_0(x) \equiv 0$ (see Lemma 1.6). When $f(u) = \lambda u^p + u^q$ with $0 < p < 1 < q$ and $u_0(x) \geq 0$, the asymptotic behavior of positive solutions is investigated in [7] and moreover the comparison theorem for positive solutions is proved. There are many contributions for (1.1) with the sublinear function $f(u) = |u|^{p-1}u$ or the concave–convex function $f(u) = |u|^{p-1}u + |u|^{q-1}u$ with $0 < p < 1 < q$ (see [1,2,9–11,14,21,22]). It seems to the author that almost all the papers deal with the power nonlinearity only.

The purpose of the present paper is to investigate the stability of stationary solutions for more general sublinear functions $f(x,u)$ as well as the power nonlinearity. The main results, stability and instability of stationary solutions, will be proved by using the method developed in the paper [4]. However, we use a novel technique to prove the existence of solutions for (1.1), the compactness of the orbit of solutions, the instability of small solutions, the exponential stability of a unique positive stationary solution and the instability of sign-changing solutions in one space dimension.

The stationary problem is written as

$$-\Delta v = f(x,v) \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Let λ_1 be the first eigenvalue of $-\Delta$ and let $\phi_1(x)$ be the corresponding eigenfunction, that is,

$$-\Delta\phi_1 = \lambda_1\phi_1, \quad \phi_1 > 0 \quad \text{in } \Omega, \qquad \phi_1 = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Assumption 1.1. We assume the following conditions:

(f1) $f(x,u)$ is continuous on $\overline{\Omega} \times \mathbb{R}$ and odd with respect to $u \in \mathbb{R}$, i.e., $f(x,-u) = -f(x,u)$ and locally Hölder continuous with respect to u, that is, for any $M > 0$, there exist constants $C > 0$ and $\sigma \in (0,1)$ such that

$$|f(x,u) - f(x,v)| \leq C|u-v|^\sigma \quad \text{for } |u|, |v| \leq M, \ x \in \overline{\Omega};$$

(f2) there exist constants $C > 0$ and $p > 1$ such that

$$|f(x,u)| \leq C(|u|^p + 1) \quad \text{for } u \in \mathbb{R}, \ x \in \overline{\Omega},$$

where $1 < p < \infty$ if $N = 1, 2$ and $1 < p < N/(N-2)$ if $N \geq 3$;

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