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Elliptic operators with unbounded diffusion, drift and potential terms ☆

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Abstract

We prove that the realization A_p in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of the elliptic operator $A = (1 + |x|^\alpha)\Delta + b|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla - c|x|^\beta$ with domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N)\}$ generates a strongly continuous analytic semigroup $T(\cdot)$ provided that $\alpha > 2$, $\beta > \alpha - 2$ and any constants $b \in \mathbb{R}$ and $c > 0$. This generalizes the recent results in [4] and in [16]. Moreover we show that $T(\cdot)$ is consistent, immediately compact and ultracontractive.

MSC: 47D07; 47D08; 35J10; 35K20

Keywords: One-parameter semigroups; Elliptic operators with unbounded coefficients; Schrödinger operator

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1. Introduction

Starting from the 1950's, the theory of linear second order elliptic operators with bounded coefficients has widely been studied. In recent years there has been a surge of activity focused on the case of unbounded coefficients. Let us recall some recent results concerning elliptic operators having polynomial coefficients.

In this paper, we are interested in studying quantitative and qualitative properties in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of the elliptic operator

$$Au(x) = q(x)\Delta u(x) + F(x) \cdot \nabla u - V(x)u(x), \quad x \in \mathbb{R}^N, \tag{1}$$

where $q(x) = (1 + |x|^\alpha)$, $F(x) = b|x|^{\alpha-2}x$, $V(x) = c|x|^\beta$, $b \in \mathbb{R}$ and $c > 0$, in the case $\alpha > 2$ and $\beta > \alpha - 2$.

Let us denote by L the operator A with $c = 0$ and illustrate the difference between the case $\alpha \in [0, 2]$ and $\alpha > 2$.

If $\alpha \in [0, 2]$ (after a modification of the drift term F near the origin, when $\alpha < 2$), it is proved in [8] that the L^p-realization L_p of L generates an analytic semigroup in $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$. Moreover, if $1 < p < \infty$, then

$$D(L_p) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) : (1 + |x|^\alpha)^{1/2}|\nabla u|,\ (1 + |x|^\alpha)|D^2u| \in L^p(\mathbb{R}^N)\}.$$

The proof of the above result is essentially based on the a-priori estimates

$$\|(1 + |x|^\alpha)^{1/2}\nabla u\|_p \le C(\|Lu\|_p + \|u\|_p)$$
$$\|(1 + |x|^\alpha)D^2u\|_p \le C(\|Lu\|_p + \|u\|_p)$$

for $u \in C_c^\infty(\mathbb{R}^N)$.

The picture changes drastically when $\alpha > 2$. In this case G. Metafune et al. in [16] showed, if $\frac{N}{N-2+b} < p < \infty$, the generation of an analytic semigroup in $L^p(\mathbb{R}^N)$ which is contractive if and only if $p \ge \frac{N+\alpha-2}{N-2+b}$. Domain characterization and spectral properties as well as kernel estimates have been also proved.

Here the techniques are based on proving some bounds on the Green function associated to the operator L.

In [11] (resp. [4]) the generation of an analytic semigroup of the L^p-realization of the Schrödinger-type operators $(1 + |x|^\alpha)\Delta - |x|^\beta$ in $L^p(\mathbb{R}^N)$ for $\alpha \in [0, 2]$ and $\beta > 2$ (resp. $\alpha > 2$, $\beta > \alpha - 2$) is obtained. In [11,5] some estimates for the associated heat kernel are provided. Also in this case the methods for $\alpha \in [0, 2]$ and $\alpha > 2$ are completely different. This is related essentially to the fact that generation of a semigroup in $L^p(\mathbb{R}^N)$ in the case $\alpha > 2$ of the operator $(1 + |x|^\alpha)\Delta$ depends upon N, see [14], [15] and does not depend if $\alpha \le 2$, see [17]

More recently in [12] the authors showed that the operator $L = |x|^\alpha\Delta + b|x|^{\alpha-2}x \cdot \nabla - c|x|^{\alpha-2}$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ if and only if $s_1 + \min\{0, 2-\alpha\} < \frac{N}{p} < s_2 + \max\{0, 2-\alpha\}$, where s_i are the roots of the equation $c + s(N-2+b-s) = 0$. Moreover the domain of the generator is also characterized.

At this point it is important to note that the techniques used in [12] are completely different from ours and lead to results which are not comparable with our case ($\beta > \alpha - 2$).