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Optimal solvability for a nonlocal problem at critical growth ☆

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Abstract

We provide optimal solvability conditions for a nonlocal minimization problem at critical growth involving an external potential function a. Furthermore, we get an existence and uniqueness result for a related nonlocal equation.

MSC: primary 35R11, 35J92, 35B33; secondary 35A15

Keywords: Brezis–Nirenberg problem; Critical growth problems; Minimization problems

Contents

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1. Introduction

1.1. Overview

Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 3$. In 1983, in the celebrated paper [5], Brezis and Nirenberg studied the solvability conditions for the semi-linear elliptic problem

$$\begin{cases} -\Delta u - \lambda\, u = u^{(N+2)/(N-2)}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In particular, if $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet–Laplacian in Ω, they proved that, if $N \geq 4$, then problem (1.1) admits a solution if $0 < \lambda < \lambda_1(\Omega)$ while for $N = 3$ there exists $\lambda^* \in (0, \lambda_1)$ such that (1.1) admits a solution if $\lambda^* < \lambda < \lambda_1(\Omega)$ and no solution for $0 < \lambda \leq \lambda^*$. Due to this phenomenon, $N = 3$ is often referred to in the literature as *critical dimension*.

In general λ^* is not given explicitly, except when Ω is a ball, in which case $\lambda^* = \lambda_1(\Omega)/4$. In addition, there is no solution to (1.1) when $\lambda \geq \lambda_1(\Omega)$ for any domain Ω (see [5, Remark 1.1]) and also for $\lambda \leq 0$ provided Ω is smooth and star-shaped (see [5, Remark 1.2]).

In the same paper the authors considered, for $N \geq 4$, the non-autonomous critical elliptic problem

$$\begin{cases} -\Delta u + a\, u = u^{(N+2)/(N-2)}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and obtained the existence of a solution by assuming that $a \in L^\infty(\Omega)$ and that there exist $\delta > 0$ and an open subset $\Omega_0 \subset \Omega$ such that

$$a \leq -\delta, \quad \text{in } \Omega_0, \qquad \int_\Omega \left(|\nabla\varphi|^2 + a\,\varphi^2\right) dx \geq \delta \int_\Omega \varphi^2 dx, \qquad \text{for all } \varphi \in C_0^\infty(\Omega),$$

see [5, Section 4]. About the case of the critical dimension $N = 3$ for (1.2), no result is stated in [5].

After the striking achievements of [5], many works were devoted to the search of solvability conditions for (possibly sign-changing) solutions of the problem

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