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On the Boussinesq–Burgers equations driven by dynamic boundary conditions

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Abstract

We study the qualitative behavior of the Boussinesq–Burgers equations on a finite interval subject to the Dirichlet type dynamic boundary conditions. Assuming $H^1 \times H^2$ initial data which are compatible with boundary conditions and utilizing energy methods, we show that under appropriate conditions on the dynamic boundary data, there exist unique global-in-time solutions to the initial-boundary value problem, and the solutions converge to the boundary data as time goes to infinity, regardless of the magnitude of the initial data.

MSC: 35Q35; 35Q51

Keywords: Boussinesq–Burgers equations; Initial-boundary value problem; Dynamic boundary conditions; Global well-posedness; Long-time behavior

1. Introduction

1.1. Background

A tidal bore is a tidal phenomenon occurring in a river or narrow bay in which the propagation of waves is against the direction of the river or bay's current. It usually appears when a shock

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wave advances into still water. Scientifically, tidal bores are classified into two categories: strong bores which are characterized by intense turbulence and turbulent mixing generated during propagation, and weak or undular bores which contain a smooth wavefront followed by a train of secondary waves known as whelps. Because of its turbulent nature, the mathematical study of strong bores is significantly challenging, due to the modeling equations must reflect the wave breaking phenomenon associated with such waves. On the other hand, weak or undular bores are much more approachable, since they usually exhibit smooth and solitary-like behavior.

Mathematical modeling of undular bores was initiated more than two centuries ago. Among the classical models in water wave theory, the Korteweg–de Vries (K–dV) equation [10,25]:

$$\partial_t v + 6v\partial_x v + \partial_{xxx} v = 0, \tag{1.1}$$

is one of the prototypes for describing the one-way (unidirectional) propagation of undular bores, whose study has inspired the development of important analytical methods in modern mathematics, such as the inverse scattering theory [18,19]. On the contrary, despite their wider range of potential applicability, mathematical understanding of the two-way models, such as the classical Boussinesq system [9]:

$$\begin{cases} \partial_t \rho + \partial_x v + \partial_x(\rho v) = 0, \\ \partial_t v + \partial_x \rho + v\partial_x v = -\dfrac{1}{3}\partial_{xtt}\rho, \end{cases} \tag{1.2}$$

or its regularization [44]:

$$\begin{cases} \partial_t \rho + \partial_x v + \partial_x(\rho v) = 0, \\ \partial_t v + \partial_x \rho + v\partial_x v = \dfrac{1}{3}\partial_{xxt} v, \end{cases} \tag{1.3}$$

is much less satisfactory than the unidirectional models.

Mathematically, both the K–dV equation (1.1) and the classical Boussinesq system (1.2) or its regularization (1.3) incorporate the nonlinear and dispersive effects, which are two of the essential features of weak bore propagation. However, early experimental results (cf. [7,20,21]) had shown that in addition to nonlinearity and dispersion, the effect of dissipation must be included in the modeling equations, in order to accurately predict the behavior of bore propagation, at least on the laboratory scale.

The object of this paper is the so-called Boussinesq–Burgers system:

$$\begin{cases} \partial_t u + \partial_x(uv) = \varepsilon\partial_{xx} u, \\ \partial_t v + \partial_x\left(u + \dfrac{v^2}{2}\right) = \mu\partial_{xx} v + \delta\partial_{xxt} v, \end{cases} \tag{1.4}$$

which can be regarded as a viscous perturbation of the regularized version of the classical Boussinesq system (1.3). It is also an appended version, by adding viscosity to both the equations, of one specific member of the well-known Boussinesq-*abcd* system, describing the two-way propagation of small amplitude gravity waves on the surface of water in a canal, derived by Bona, Chen and Saut [5]:

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