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Renormalized radial large-data solutions to the higher-dimensional Keller–Segel system with singular sensitivity and signal absorption

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Abstract

The chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot \left(\frac{u}{v}\nabla v\right), \\ v_t = \Delta v - uv, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in the ball $\Omega = B_R(0) \subset \mathbb{R}^n$, where $R > 0$ and $n \geq 2$.

Despite its great relevance as a model for the spontaneous emergence of spatial structures in populations of primitive bacteria, since its introduction by Keller and Segel in 1971 this system has been lacking a satisfactory theory even at the level of the basic questions from the context of well-posedness; global existence results in the literature are restricted to spatially one- or two-dimensional cases so far, or alternatively require certain smallness hypotheses on the initial data.

For all suitably regular and radially symmetric initial data (u_0, v_0) satisfying $u_0 \geq 0$ and $v_0 > 0$, the present paper establishes the existence of a globally defined pair (u, v) of radially symmetric functions which are continuous in $(\bar{\Omega} \setminus \{0\}) \times [0, \infty)$ and smooth in $(\bar{\Omega} \setminus \{0\}) \times (0, \infty)$, and which solve the corresponding initial-boundary value problem for $(\star)$ with $(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0)$ in an appropriate generalized sense. To the best of our knowledge, this in particular provides the first result on global existence for the three-dimensional version of $(\star)$ involving arbitrarily large initial data.

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1. Introduction

This work is concerned with the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot \left(\frac{u}{v}\nabla v\right), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^n$, where our main focus will be on the case when $\Omega = B_R(0) \subset \mathbb{R}^n$ with $R > 0$ and $n \geq 2$, and on solutions which are radially symmetric with respect to $|x|$ and nonnegative in their first and positive in their second component.

The PDE system in (1.1) has been proposed by Keller and Segel in the second of their seminal works ([14]) in order to describe the spatio-temporal behavior in populations of cells which, besides diffusing randomly, partially orient their movement toward increasing concentrations of a diffusible signal substance which they consume upon contact; in this framework, $u = u(x,t)$ denotes the density of cells and $v = v(x,t)$ represents the nutrient concentration (cf. [23], [17], [13], [26] and [18] for further modeling aspects, also in different biological contexts). As indicated by formal and numerical as well as rigorous analytical results on existence and stability properties of wave-like solutions ([12], [21], [20], [9]), the interplay of chemotaxis and absorption mechanisms as modeled by (1.1) is indeed able to support the emergence of spatially heterogeneous structures, as known from experimental observations to be a striking feature of such simple biological settings ([1]).

Despite its evident relevance in biological applications, a comprehensive solution theory for (1.1) seems yet lacking. In fact, the singular behavior near $v = 0$ of the chemotactic sensitivity function, in (1.1) chosen as $S(u,v) = \frac{u}{v}$, may considerably enhance the relative strength of cross-diffusion at each point where the signal concentration becomes small, and the second equation in (1.1) suggests to conjecture that the set of such points should become substantially large during evolution. Accordingly, it is still an open problem to decide whether in spatially higher-dimensional cases, (1.1) may enforce the spontaneous formation of singularities, as known to occur e.g. in the classical Keller–Segel system obtained on choosing $S(u) = u$ as chemotactic sensitivity, and replacing the second equation in (1.1) by the equation $v_t = \Delta v - v + u$ modeling signal production by cells (see [10], [34] and also [3] for a survey on this and related problems). In presence of non-singular chemotactic cross-diffusion determined by the choice $S(u) = u$, the dissipative action due to a signal absorption mechanism as in (1.1) is known to entirely suppress such blow-up phenomena at least in two-dimensional situations, and in the case $n = 3$ at least certain global weak solutions can always be constructed ([29]), being classical whenever v is suitably small ([28], [38]). However, the destabilizing potential of singular sensitivities of the form in (1.1) is far from understood even in systems which account for signal production and hence counteract the tendency of the quantity v to attain small values; for such systems in the case $n \geq 2$, namely, global existence and boundedness of classical solutions is guaranteed