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# Intermittency for stochastic partial differential equations driven by strongly inhomogeneous space–time white noises

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**Abstract**

In this paper, the main topic is to investigate the intermittent property of the one-dimensional stochastic heat equation driven by an inhomogeneous Brownian sheet, which is a noise deduced from the study of the catalytic super-Brownian motion. Under some proper conditions on the catalytic measure of the inhomogeneous Brownian sheet, we show that the solution is weakly full intermittent based on the estimates of moments of the solution. In particular, it is proved that the second moment of the solution grows at the exponential rate. The novelty is that the catalytic measure relative to the inhomogeneous noise is not required to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.


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*Keywords:* Stochastic partial differential equation; Inhomogeneous noise; Lyapunov exponent; Intermittency; Noise excitation

## 1. Introduction and main results

We consider the intermittency of the following one-dimensional stochastic partial differential equation (SPDE, for abbreviation) driven by an inhomogeneous space–time white noise:

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$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\frac{\partial^2}{\partial t\partial x}w_\rho(t,x),\ t>0, x\in\mathbb{R}, \\ u(0,x)=u_0(x),\ x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where the coefficient $\sigma$ is a non-random measurable function defined on $\mathbb{R}$ and $w_\rho(t,x)$ is called the inhomogeneous two-parameter Brownian sheet $[0,\infty)\times\mathbb{R}$ with its intensity measure $dt\rho(dx)$. Hereafter, $\rho$ denotes a $\sigma$-finite positive Radon measure on $\mathbb{R}$, which is possible to be singular with respect to the Lebesgue measure on $\mathbb{R}$. For the precise meaning of the noise $\frac{\partial^2}{\partial t\partial x}w_\rho(t,x)$ in (1.1) and the mathematical definition of this equation, we refer the reader to Section 2.

This kind of SPDEs includes the parabolic Anderson model (as $\sigma(u)=u$, see [1] or [6] for detailed discussion) as its important example, and is also closely connected to the stochastic Burgers equation [2,24] and the Kardar–Parisi–Zhang equation [2,23,25]. Hence, a lot of interesting analytic properties, such as the weak intermittency of the solution and the noise excitation, are actively studied, see [7], [20], [29] and references therein.

The main purpose of this paper is devoted to the study of some important properties, especially, the intermittent property of the stochastic heat equation (1.1) driven by the inhomogeneous space–time white noise. The intermittency formally means that the random field exhibits some sharp peaks, which has the main contribution to its statistical moments. We refer the reader to [1] or [27] for the detailed explanation of the meaning of the intermittency relative to SPDEs. It is known that the intermittent property is very important not only in mathematics but also in physics. So, after the groundbreaking work [1], there are many research papers being published. In 1995, L. Bertini and N. Cancrini [1] investigated the intermittency of the parabolic Anderson model on the real line $\mathbb{R}$, that is, the SPDE (1.1) with $\sigma(u)=u$ and $\rho(dx)=dx$ by the Feynman–Kac formula. In 2009, M. Foondun and D. Khoshnevisan [20] started to study the nonlinear SPDEs driven by homogeneous space–time white noises. To do that, they generalized the definition of intermittency to a weak one, see Definition 2.3 in Section 2. After that, the research on the weak intermittency of one-dimensional SPDEs attracts many authors' attention and different approaches are introduced, see [7,9,17,18,21,28,29], the monograph [27] for SPDEs on the whole space and respectively [19,22,36] for SPDEs on bounded domains.

There are two motivations to investigate the stochastic heat equation (1.1). The first is the close connection with the study of the catalytic super-Brownian motion, which is a class of stochastic processes interacting with a catalytic medium and is initially introduced by D.A. Dawson and K. Fleischmann [11].

The study of the catalytic super-Brownian motion has attracted considerable interest after the important survey by Klenke [30], for example, see [3,13,15,16,33]. Heuristically speaking, the catalytic super-Brownian motion is a continuous inhomogeneous measure-valued branching process, which is the high-density/short-lifetime limit of critical binary branching Brownian particles with the varying branching intensity characterized by a catalyst $dt\rho(dx)$ in space and time, which is also called the intensity measure in this paper. If $\rho(dx)=dx$, then it is just the classical super-Brownian motion or the Dawson–Watanabe process with a constant branching rate. In this special case, it is well-known that the Dawson–Watanabe process has a jointly continuous density field on $[0,\infty)\times\mathbb{R}$, which is the characterized by the SPDE (1.1) with the homogeneous space–time white noise and $\sigma(u)=\sqrt{u}$, see [31]. However, for the catalytic super-Brownian motion, it is known that if the catalyst $\rho$ is too rough, then it does not have jointly continuous density field on the whole space $[0,\infty)\times\mathbb{R}$. For example, if $\rho(dx)=\delta_a(dx)$ for some constant $a\in\mathbb{R}$, the density of the catalytic super-Brownian motion blows up at the catalytic position $a$,