ARTICLE IN PRESS YJDEQ:9073

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

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Boundary regularity criteria for the 6D steady Navier–Stokes and MHD equations

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Received 3 December 2016; revised 16 October 2017

Abstract

It is shown in this paper that suitable weak solutions to the 6D steady incompressible Navier–Stokes and MHD equations are Hölder continuous near boundary provided that either $r^{-3}\int_{B_r^+}|u(x)|^3dx$ or $r^{-2}\int_{B_r^+}|\nabla u(x)|^2dx$ is sufficiently small, which implies that the 2D Hausdorff measure of the set of singular points near the boundary is zero. This generalizes recent interior regularity results by Dong–Strain [5].

MSC: 35Q30; 76D03

Keywords: Navier–Stokes equations; MHD equations; Suitable weak solutions; Boundary regularity

1. Introduction

In this paper, we consider the following 6D steady incompressible Navier–Stokes equations on $\Omega\subset\mathbb{R}^6$:

$$\text{(SNS)}\quad \begin{cases} -\Delta u + u\cdot\nabla u = -\nabla\pi + f, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

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https://doi.org/10.1016/j.jde.2017.10.028

where u represents the fluid velocity field, π is a scalar pressure, and the boundary condition of u is given as a no-slip condition, namely

$$u = 0, \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

The analysis of the above equations is motivated by Struwe's question in [35], where the 5D steady Navier–Stokes equations was considered and he asked if analogous partial regularity results hold in spacial dimension $N > 5$. Recently interior regularity results in 6D are obtained by Dong–Strain [5], and the main interests in this paper are in the boundary partial regularity for suitable weak solutions to the equations (1.1). As it's commented in [5], six is the highest dimension for stationary Navier–Stokes equations in which all the existing methods on partial regularity could be applied.

Recall the development of interior and boundary regularity criteria for the Navier–Stokes equations in brief. For the three dimensional time-dependent Navier–Stokes equations, partial regularity of weak solutions satisfying the local energy inequality was proved in a series of papers by Scheffer [30,31,33]. Later, the notion of suitable weak solutions was first introduced in a celebrated paper by Caffarelli–Kohn–Nirenberg [1], showing that the set $\mathcal{S}$ of possible interior singular points of a suitable weak solution is one-dimensional parabolic Hausdorff measure zero. Simplified proofs and improvements can be found in many works by Lin [22], Ladyzhenskaya–Seregin [23], Tian–Xin [37], Escauriaza–Seregin–Šverák [6], Seregin [26], Gustafson–Kang–Tsai [18], Vasseur [40], Kukavica [21], Wang–Zhang [39] and the references therein. Some similar boundary regularity results are also proved, see Seregin [25,27, 28], Kang [19,20], Gustafson–Kang–Tsai [17] and so on. Moreover, it's worth to mention that general curved boundary regularity was obtained by Seregin–Shilkin–Solonnikov in [29].

There are only few results available in the literature for the 4D and higher dimensional time-dependent Navier–Stokes equations. In [32], Scheffer showed that there exists a weak solution in $R^4 \times R^+$, whose singular set has vanishing 3D Hausdorff measure. Later, Dong–Du [2] proved that, for any local-in-time smooth solution to the 4D Navier–Stokes equations, the 2D Hausdorff measure of the set of singular points at the first potential blow-up time is equal to zero, and we refer to [3] for recent results with general suitable weak solutions.

Now we turn to the steady Navier–Stokes equations. In a series of papers by Frehse and Ruzicka [7–10], the existence on a class of special regular solutions of (1.1) was obtained for the five-dimensional and higher dimensional case. Gerhardt [13] obtained the regularity of weak solutions under the four-dimensional case. For $N \geq 5$, it is not known yet whether weak solutions are regular. In [35,36], Struwe obtained partial regularity for $N = 5$ by regularity methods of elliptic systems (cf. Morrey [24] and Giaqinta [14]). Later, the result of Struwe was extended to the boundary case by Kang [19]. In a recent paper, Dong–Strain [5] extended the interior regularity result of Struwe to the six-dimensional space. Their main idea is to apply the iteration method and bootstrap arguments to get a suitable decay estimate of $L^{3/2}$ norm of ∇u, then the Morrey lemma implies the required regularity.

In this paper, we generalized the result in [5] and proved the boundary regularity of six-dimensional steady Navier–Stokes equations. Moreover, we considered six-dimensional steady MHD equations, and obtained boundary regularity criteria independent of the magnetic field. It is important to mention that H. Dong et al. [4] also obtained boundary regularity criteria for the 6D steady-state case independently, and Theorem 1.4 in [4] is similar to the following Theorem 1.2 by a different approach. Moreover, boundary regularity criteria for the 4D time-