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Global uniqueness in an inverse problem for time fractional diffusion equations

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Abstract

Given (M, g), a compact connected Riemannian manifold of dimension $d \geqslant 2$, with boundary ∂M, we consider an initial boundary value problem for a fractional diffusion equation on $(0, T) \times M$, $T > 0$, with time-fractional Caputo derivative of order $\alpha \in (0, 1) \cup (1, 2)$. We prove uniqueness in the inverse problem of determining the smooth manifold (M, g) (up to an isometry), and various time-independent smooth coefficients appearing in this equation, from measurements of the solutions on a subset of ∂M at fixed time. In the "flat" case where M is a compact subset of $\mathbb{R}^d$, two out the three coefficients ρ (density), a (conductivity) and q (potential) appearing in the equation $\rho\partial_t^\alpha u - \text{div}\,(a\nabla u) + qu = 0$ on $(0, T) \times M$ are recovered simultaneously.

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Keywords: Inverse problems; Fractional diffusion equation; Partial data; Uniqueness result

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1. Introduction

1.1. Statement of the problem

Let (M, g) be a compact connected Riemannian manifold of dimension $d \geqslant 2$, with boundary ∂M. For a strictly positive function μ we consider the weighted Laplace–Beltrami operator

$$\Delta_{g,\mu} := \mu^{-1}\mathrm{div}_g\, \mu\, \nabla_g,$$

where div_g (resp., ∇_g) denotes the divergence (resp., gradient) operator on (M, g), and $\mu^{\pm 1}$ stands for the multiplier by the function $\mu^{\pm 1}$. If μ is identically 1 in M then $\Delta_{g,\mu}$ coincides with the usual Laplace–Beltrami operator on (M, g). In local coordinates, we have

$$\Delta_{g,\mu} u = \sum_{i,j=1}^{d} \mu^{-1}|g|^{-1/2}\partial_{x_i}(\mu|g|^{1/2}g^{ij}\partial_{x_j}u), \quad u \in C^\infty(M),$$

where $g^{-1} := (g^{ij})_{1\leqslant i,j\leqslant d}$ and $|g| := \det g$. For $\alpha \in (0, 2)$ we consider the initial boundary value problem (IBVP)

$$\begin{cases} \partial_t^\alpha u - \Delta_{g,\mu}u + qu &= 0, \quad \text{in } (0,T)\times M, \\ u &= f, \quad \text{on } (0,T)\times \partial M, \\ \partial_t^k u(0,\cdot) &= 0, \quad \text{in } M,\ k = 0, \dots, m, \end{cases} \tag{1.1}$$

with non-homogeneous Dirichlet data f. Here $m := [\alpha]$ denotes the integer part of α and ∂_t^α is the Caputo fractional derivative of order α with respect to t, defined by

$$\partial_t^\alpha u(t,x) := \frac{1}{\Gamma(m+1-\alpha)} \int_0^t (t-s)^{m-\alpha}\partial_s^{m+1}u(s,x)\mathrm{d}s, \ (t,x) \in Q, \tag{1.2}$$

where Γ is the usual Gamma function expressed as $\Gamma(z) := \int_0^{+\infty} e^{-t}t^{z-1}\mathrm{d}t$ for all $z \in \mathbb{C}$ such that $\Re z > 0$. The system (1.1) models anomalous diffusion phenomena. In the sub-diffusive case $\alpha \in (0, 1)$, the first line in (1.1) is usually named fractional diffusion equation, while in the super-diffusive case $\alpha \in (1, 2)$, it is referred as fractional wave equation.

Given two nonempty open subsets S_{in} and S_{out} of ∂M, $T_0 \in (0, T)$ and $\alpha \in (0, 2)$, we introduce the function space

$$\mathcal{H}_{\text{in},\alpha,T_0} := \{f \in C^{[\alpha]+1}([0,T], H^{\frac{3}{2}}(\partial M));\ \mathrm{supp}\, f \subset (0,T_0)\times S_{\text{in}}\},$$

where we recall that $[\alpha]$ stands for the integer part of α. As established in Section 2, problem (1.1) associated with $f \in \mathcal{H}_{\text{in},\alpha,T_0}$ is well posed and the partial Dirichlet-to-Neumann (DN) map

$$\Lambda_{M,g,\mu,q} : \mathcal{H}_{\text{in},\alpha,T_0} \ni f \mapsto \partial_\nu u(T_0,\cdot)_{|S_{\text{out}}} := \sum_{i,j=1}^{d} g^{ij}\nu_i\partial_{x_j}u(T_0,\cdot)_{|S_{\text{out}}}, \tag{1.3}$$

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