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Local kinetic energy and singularities of the incompressible Navier–Stokes equations

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Abstract

We study the partial regularity problem of the incompressible Navier–Stokes equations. A reverse Hölder inequality of velocity gradient with increasing support is obtained under the condition that a scaled functional corresponding the local kinetic energy is uniformly bounded. As an application, we give a new bound for the Hausdorff dimension and the Minkowski dimension of singular set when weak solutions v belong to $L^\infty(0, T; L^{3,w}(\mathbb{R}^3))$ where $L^{3,w}(\mathbb{R}^3)$ denotes the standard weak Lebesgue space.

MSC: 35K20; 35B50

Keywords: Navier–Stokes equations; Partial regularity; Minkowski dimension; Singular set

1. Introduction

In this paper we study the partial regularity problem of the incompressible Navier–Stokes equations. Although general boundary and geometric conditions are important, we consider merely an initial value problem of the Navier–Stokes equations in the whole space $\Omega_T = \mathbb{R}^3 \times (0, T)$:

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$$\begin{aligned}(\partial_t - \nu\Delta)v + \operatorname{div}(v\otimes v) + \nabla p = 0\\ \operatorname{div} v = 0\end{aligned} \tag{1}$$

with the initial data $v(x,0) = v_0(x)$, where v and v_0 are three dimensional solenoidal vector fields and the pressure p is a scalar field. In this paper, we let the viscosity $\nu = 1$ since it is not important in our regularity analysis. We denote by $z = (x,t)$ space–time points, space balls by $B(x,r) = \left\{y \in \mathbb{R}^3 : |y-x| < r\right\}$, and parabolic cylinders by

$$Q(z,r) = B(x,r) \times (t-r^2, t+r^2).$$

We always assume that parabolic cylinders are in the space–time domain and suppress reference points z in various expressions when it can be understood obviously in the context. The precise definitions will be given in the next section.

It is not known whether the solution stays regular for all time although the global existence of weak solutions was proved by Leray [20] long ago. To study the regularity problem Scheffer [24] introduced the concept of suitable weak solutions and proved a partial regularity result. From Scheffer's result, one can conclude that the Minkowski dimension of the singular set is not greater than 2. Caffarelli–Kohn–Nirenberg [1] proved that the Hausdorff dimension of the singular set is not greater than 1 using a regularity criterion based on a scaled invariant functional corresponding to the velocity gradient. Lin [21] gave a simplified short proof by a blowup argument. Ladyzhenskaya and Seregin [19] gave a clear presentation of the Hölder regularity. Choe and Lewis [3] studied the singular set by using a generalized Hausdorff measure. Gustafson, Kang and Tsai [11] unified several known regularity criteria. One of the most important conditions to guarantee the regularity of weak solutions is the so-called Ladyzhenskaya–Prodi–Serrin (LPS) [18,22,28] condition, that is,

$$u \in L^l(0,T; L^s(\mathbb{R}^3))$$

for some s and l satisfying $\frac{3}{s} + \frac{2}{l} = 1$ and $3 < s \le \infty$. Escauriaza–Seregin–Šverák [5] resolved the regularity problem for the marginal case $v \in L^\infty(0,T; L^3(\mathbb{R}^3))$. There are many variations and generalizations of the LPS condition including the Lorentz spaces. But, most of them dealt with the case that the space integrability exponent is greater than 3. The marginal case $v \in L^\infty(0,T; L^{3,w}(\mathbb{R}^3))$ of the Lorentz spaces was studied by Kozono [15] and Kim–Kozono [12], where $L^{3,w}(\mathbb{R}^3)$ denotes the standard weak Lebesgue space. They obtained some regularity results when the norm $\|v\|_{L^\infty(0,T;L^{3,w}(\mathbb{R}^3))}$ is sufficiently small. Recently, there arise many interests on this much weaker case $v \in L^\infty(0,T; L^{3,w}(\mathbb{R}^3))$, without any smallness condition, related to type I singularities.

In studying the regularity problem of the Navier–Stokes equations, many papers dealt with the quantities in terms of gradients of solutions like $\int_Q |\nabla v|^2$ or $\int_Q |\omega|^2$ where ω denotes the vorticity $\omega = \nabla \times v$. In this paper, we study the regularity problem based on local kinetic energy $\int_{B(x,R)} |v(y,t)|^2 dy$. Under the condition that a scaled functional corresponding the local kinetic energy is uniformly bounded, we show that a reverse Hölder inequality of ∇v with increasing support holds. Here is our first theorem.

Theorem 1. *Suppose there exists a number $M \ge 1$ such that for all $Q(z,R) \subset \Omega_T$*

$$\sup_{|s-t|<R^2} R^{-1} \int_{B(x,R)} |v(y,s)|^2 dy \le M. \tag{2}$$