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On the well-posedness of 3-D inhomogeneous incompressible Navier–Stokes equations with variable viscosity

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Abstract

In this paper, we mainly study the well-posedness for the 3-D inhomogeneous incompressible Navier–Stokes equations with variable viscosity. With some smallness assumption on the BMO-norm of the initial density, we first get the local well-posedness of (1.1) in the critical Besov spaces. Moreover, if the viscosity coefficient is a constant, we can extend this local solution to be a global one. Our theorem implies that we have successfully extended the integrability index p of the initial velocity which has been obtained by Abidi, Gui and Zhang in [3], Burtea in [8] and Zhai and Yin in [32] to approach the ideal one i.e. $1 < p < 6$. The main novelty of this work is to apply the CRW theorem obtained by Coifman, Rochberg, Weiss in [11] to get a new a priori estimate for an elliptic equation with variable coefficients. The uniqueness of the solution also relies on a Lagrangian approach as in [16–18].

MSC: 35Q35; 35Q30; 76D03

Keywords: Well-posedness; Navier–Stokes equations; Besov spaces

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1. Introduction

In this paper, we study the Cauchy problem for the 3-D inhomogeneous incompressible Navier–Stokes equations with variable viscosity in critical Besov spaces

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\big(2\mu(\rho)\mathcal{M}(u)\big) + \nabla\Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

In the above system, $\rho > 0$ stands for the density of the fluid, u is the fluid's velocity field while Π is a scalar pressure function, $\mathcal{M}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, the viscosity coefficient μ is smooth, positive on $[0, \infty)$.

Such system describes a fluid that is incompressible but has non-constant density. Basic examples are mixture of incompressible and non reactant flows, flows with complex structure (e.g. blood flow or model of rivers), fluids containing a melted substance, etc. One may check [23] for the detailed derivation of this system.

There is a very rich literature dedicated to the study the above system. In the case of smooth data with no vacuum, Ladyženskaja and Solonnikov first addressed in [22] the question of unique solvability of (1.1) with constant viscosity. Similar results were obtained by Danchin [14] in $\mathbb{R}^n$ with initial data in the almost critical Sobolev spaces. Global weak solutions with finite energy were constructed by Simon in [28] (see also the book by Lions [23] for the variable viscosity case). Yet the regularity and uniqueness of such weak solutions are big open problems. Recently, Danchin and Mucha [17] proved by using a Lagrangian approach that the system (1.1) with constant viscosity has a unique local solution with initial data $(\rho_0, u_0) \in L^\infty(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ if initial vacuum does not occur. For more results on the classical solution, one can see [5,19,23] and references therein.

When the density ρ is away from zero, we can use $a = \frac{1}{\rho} - 1$ to change (1.1) into

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - (1+a)\operatorname{div}\big(2\widetilde{\mu}(a)\mathcal{M}(u)\big) + (1+a)\nabla\Pi = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0), \end{cases} \tag{1.2}$$

where $\widetilde{\mu}(a) = \mu(\frac{1}{1+a})$ is a smooth function.

Recently, many mathematicians have studied the well-posedness of the system (1.1) in the critical Besov space. We say that a function space is critical if the corresponding norm is invariant under the transformations:

$$(a_\lambda, u_\lambda)(t, x) = (a(\lambda^2\cdot, \lambda\cdot), \lambda u(\lambda^2\cdot, \lambda\cdot)).$$

We can verify that the space $\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n) \times \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ is the critical space for system (1.2).

When the initial density is close enough to a positive constant, Danchin in [13] proved that if the initial data $a_0 \in \dot{B}^{\frac{n}{2}}_{2,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $u_0 \in \dot{B}^{-1+\frac{n}{2}}_{2,1}(\mathbb{R}^n)$, then the system (1.2) with constant viscosity has a unique local-in-time solution. This result was improved by Abidi in [1] in the case with general viscosity, the author proved that if initial data $a_0 \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $u_0 \in \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n)$,

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