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Nonlinear stability of Gardner breathers

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Abstract

We show that breather solutions of the Gardner equation, a natural generalization of the KdV and mKdV equations, are $H^2(\mathbb{R})$ stable. Through a variational approach, we characterize Gardner breathers as minimizers of a new Lyapunov functional and we study the associated spectral problem, through (i) the analysis of the spectrum of explicit linear systems (*spectral stability*), and (ii) controlling degenerated directions by using low regularity conservation laws.

MSC: primary 35Q51, 35Q53; secondary 37K10, 37K40

Keywords: Gardner equation; Modified KdV equation; Breather; Stability

Contents

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1. Introduction

1.1. Preliminaries

In this paper we consider the nonlinear stability of *breathers* of the Gardner equation

$$w_t + (w_{xx} + 3\mu w^2 + w^3)_x = 0, \quad \mu \in \mathbb{R}\backslash\{0\}, \qquad w(t,x) \in \mathbb{R},\ (t,x) \in \mathbb{R}^2. \tag{1.1}$$

Specifically, we present here a proof on the stability in $H^2(\mathbb{R})$ of Gardner breathers, showing that this stability is independent of the value of the parameter μ, which controls the strength of the quadratic nonlinear part or KdV term w^2, in its existence interval for *real* Gardner breathers.

The Gardner equation (1.1) is a well-known *completely integrable* model [14,1,31], with infinitely many conservation laws and well-known (long-time) asymptotic behavior of its solutions obtained with the help of the inverse scattering transform [18]. As a physical model, (1.1) describes large-amplitude internal solitary waves, showing a dynamics which can look rather different from the KdV form. On the other hand, solutions of (1.1) are invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $w(t-t_0, x-x_0)$ is also a solution. Note that (1.1) is not scaling invariant. Moreover, (1.1) is closely related to the modified Korteweg–de Vries (mKdV) equation

$$u_t + (u_{xx} + u^3)_x = 0, \qquad u(t,x) \in \mathbb{R},\ (t,x) \in \mathbb{R}^2, \tag{1.2}$$

through the search of L^∞-solutions. In fact, it is easy to see by substitution, that the following holds:

Proposition 1.1. *Let u be a solution of the mKdV equation (1.2) with a nonvanishing boundary value or condition (NVBC) $\mu \in \mathbb{R}\backslash\{0\}$ at $\pm\infty$. Then $w(t,x) := u(t, x + 3\mu^2 t) - \mu$ is a solution of the Gardner equation (1.1).*

The key characteristic of the Gardner equation (1.1) is that it contains a nonlinear part composed of a Korteweg–de Vries (KdV) quadratic term $(w^2)_x$ and a positive modified KdV (mKdV) cubic term $(w^3)_x$. The competition between this nonlinear part and the linear dispersive term w_{xxx} allows the existence of intricate soliton, multisolitons as well as exact real-valued *breather* solutions (see (1.12)). A *soliton* is a localized, moving or stationary solution which maintains its form for all time. Similarly, a *multi-soliton* is a (not necessarily) explicit solution describing the interaction of several solitons [20].

In the case of the Gardner equation (1.1), the profile of the soliton solution is slightly cumbersome, but it is still given explicitly by the formula

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