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On approximation of Ginzburg–Landau minimizers by $\mathbb{S}^1$-valued maps in domains with vanishingly small holes

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Abstract

We consider a two-dimensional Ginzburg–Landau problem on an arbitrary domain with a finite number of vanishingly small circular holes. A special choice of scaling relation between the material and geometric parameters (Ginzburg–Landau parameter vs. hole radius) is motivated by a recently discovered phenomenon of vortex phase separation in superconducting composites. We show that, for each hole, the degrees of minimizers of the Ginzburg–Landau problems in the classes of $\mathbb{S}^1$-valued and $\mathbb{C}$-valued maps, respectively, are the same. The presence of two parameters that are widely separated on a logarithmic scale constitutes the principal difficulty of the analysis that is based on energy decomposition techniques.

Keywords: Ginzburg–Landau; Superconductivity; Vortex; Boundary value problem; Multiply-connected domain

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1. Introduction

The present study is motivated by the pinning phenomenon in type-II superconducting composites. Type-II superconductors are characterized by vanishing resistivity and complete expulsion of magnetic fields from the bulk of the material at sufficiently low temperatures. When the magnitude h_{ext} of an external magnetic field $\mathbf{h}_{ext}$ exceeds a certain threshold, the field begins to penetrate the superconductor along isolated vortex lines that may move, resulting in energy dissipation. This motion and related energy losses can be inhibited by pinning the lines to impurities or holes in a *superconducting composite*. Understanding the role of imperfections in a superconductor can thus be used to design more efficient superconducting materials. In what follows, we will consider a cylindrical superconducting sample containing rod-like inclusions or *columnar defects* elongated along the axis of the cylinder, so that the sample can be represented by its cross-section $\Omega \subset \mathbb{R}^2$. Then the vortex lines penetrate each cross-section at isolated points, called *vortices*.

Superconductivity is typically modeled within the framework of the Ginzburg–Landau theory [1] in terms of an order-parameter $u \in \mathbb{C}$ and the vector potential of the induced magnetic field $A \in \mathbb{R}^2$. The appearance and behavior of vortices for the minimizers of the Ginzburg–Landau functional

$$GL^\varepsilon[u, A] = \frac{1}{2}\int_\Omega |(\nabla - iA)u|^2\,dx + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2\,dx + \frac{1}{2}\int_\Omega (\operatorname{curl} A - h_{ext})^2\,dx \qquad (1)$$

have been studied, in particular, in [2,3] where the existence of two critical magnetic fields, H_{c1} and H_{c2}, was established rigorously for simply-connected domain when $\varepsilon > 0$ is small. When the external magnetic field is weak ($h_{ext} < H_{c1}$) it is completely expelled from the bulk semiconductor (Meissner effect) and there are no vortices. When the field strength is ramped up from H_{c1} to H_{c2}, the magnetic field penetrates the superconductor through an increasing number of isolated vortices while the superconductivity is destroyed everywhere, once the field exceeds H_{c2}.

The pinning phenomenon that we consider in this paper is observed in non-simply-connected domains with holes that may or may not contain another material. If a hole "pins" a vortex, the order parameter u has a nonzero winding number on the boundary of the hole. We refer to this object as a *hole vortex*. Note that degrees of the hole vortices increase in absolute value along with the strength of the external magnetic field. This situation is in contrast with the regular bulk vortices that have degree ± 1 and increase in number as the field becomes stronger.

An alternative way to model the impurities is to consider a potential term $(a(x) - |u|^2)^2$ where $a(x)$ varies throughout the sample. It was proven in [4] that the impurities corresponding to the weakest superconductivity (where $a(x)$ is minimal) pin the vortices first. This model was studied further in [5] and [6] to demonstrate the existence of nontrivial pinning patterns and in [7] to investigate the breakdown of pinning in an increasing external magnetic field, among other issues. A composite consisting of two superconducting samples with different critical temperatures was considered in [8,9] where nucleation of vortices near the interface was shown to occur.

In our model we consider a superconductor with holes, similar to the setup in [10]. In that work, the authors considered the asymptotic limits of minimizers of GL^ε as $\varepsilon \to 0$ and determined that holes act as pinning sites gaining nonzero degree for moderate but bounded magnetic

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