ARTICLE IN PRESS
YJDEQ:9042

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Normal linearization and transition map near a saddle connection with symmetric resonances

Peter De Maesschalck [a], Vincent Naudot [b], Jeroen Wynen [a,*]

[a] *Department of Mathematics, Hasselt University, Martelarenlaan 42, 3500 Hasselt, Belgium*

[b] *Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, FL 33431, United States*

Received 22 May 2017; revised 21 September 2017

Abstract

We consider a heteroclinic connection in a planar system, between two symmetric hyperbolic saddles of which the eigenvalues are resonant. Starting with a C^∞ normal form, defined globally near the connection, we normally linearize the vector field by using finitely smooth tags of logarithmic form. We furthermore define an asymptotic entry–exit relation, and we discuss on two examples how to deal with counting limit cycles near a limit periodic set involving such a connection.

MSC: 34C14; 34C20; 37C10; 37C15; 37C29

Keywords: Planar vector fields; Saddle connection; Invariant; Linearization; Poincaré map; Cyclicity

1. Introduction

There has been extensive research on bounding the number of isolated periodic orbits bifurcating from graphics (the cyclicity) in analytic planar vector fields in the context of Hilbert's 16th problem following an idea of Roussarie ([1]). Graphics are formed by a finite sequence of heteroclinic connections that together with the connected singular points topologically form a circle. For instance in [2] the authors reduce the problem of finding a uniform bound on the number of

* Corresponding author.
E-mail address: jeroen.wynen@uhasselt.be (J. Wynen).

https://doi.org/10.1016/j.jde.2017.09.042
0022-0396/

Please cite this article in press as: P. De Maesschalck et al., Normal linearization and transition map near a saddle connection with symmetric resonances, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.09.042

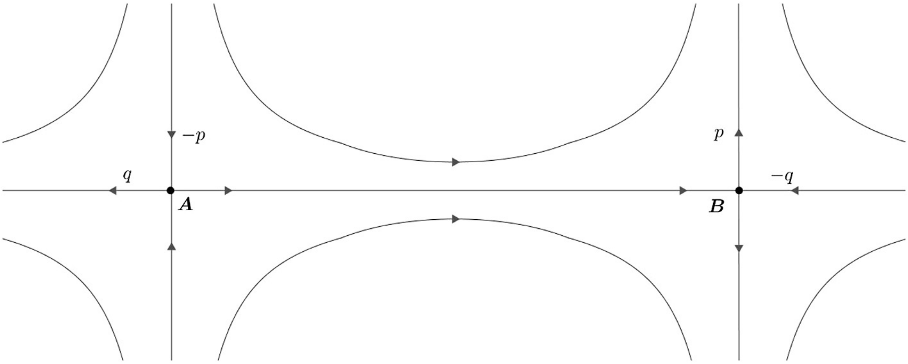

Fig. 1. Saddle connection with symmetric $q : -p$ spectrum.

limit cycles in quadratic vector fields to the study of 121 graphics. The classical way to do this is by studying the map of first return of such a graphic in order to get an upper bound. However these computations tend to be difficult in general especially in a neighbourhood of singularities. Using normal form theory (see chapter 2 of [3]) one can simplify the local calculations (e.g. near a hyperbolic saddle, see [4]). When the graphic contains non-elementary singularities, for example in cuspidal loops (see [5]), one usually uses advanced techniques like a blow-up of the vector field near the singularity.

Here we will present a tool that may be useful in dealing with graphics that contain two hyperbolic saddles. More specifically, we consider in this paper C^∞ vector fields in the plane with two hyperbolic saddles A and B having a heteroclinic connection (see Fig. 1). Without loss of generality we can assume that $A = (-1, 0)$ and $B = (1, 0)$. We impose that the linearization of the vector field about A (resp. about B) has a $-p : q$ (resp. $p : -q$) resonant spectrum; p and q positive and relatively prime integers. In this paper, we do not consider unfoldings, i.e. here we do not consider families of vector fields in which the parameters either break the saddle connection and/or perturb the ratios of eigenvalues. This setting, where the ratios of eigenvalues are fixed and the saddle connection is unbroken is often encountered when studying polynomial vector fields at ∞, or when blowing up nilpotent or degenerate singular points; the saddle connection is then found as a segment on the equator of the blow-up circle.

Under the imposed conditions a C^∞ normal form (up to time rescaling) near the connection has been obtained in [6]:

$$\begin{cases} \dot{x} = \frac{q}{2}(1 - x^2) \\ \dot{y} = y\left(px + w^n f(w) + x w^n g(w) + \chi(x) h(y)\right), \end{cases} \tag{1}$$

where $w = (1 - x^2)^p y^q$ and χ is infinitely flat at $x = \pm 1$, $n \geq 1$ and all occurring functions are C^∞. For readers familiar with local normal form theory, it might be beneficial to realize that the local normal forms about A and B have resonant terms of the form $(1 + x)^p y^q$ and $(1 - x)^p y^q$. The expression $x w^n g(w)$ represents the part of the normal form where B behaves truly reversible w.r.t. A; it is the symmetric part. The expression $w^n f(w)$ represents the anti-symmetric part. The function $\chi(x)h(y)$ contains the so-called connecting terms (terminology from [6]); it is only present when $q \neq 1$. We will see that these terms may have an effect that is distinguishably different from the effect of the resonant terms on the dynamics near the connection.

Please cite this article in press as: P. De Maesschalck et al., Normal linearization and transition map near a saddle connection with symmetric resonances, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.09.042