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# Rate of convergence for one-dimensional quasilinear parabolic problem and its applications

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**Abstract**

Based on a comparison principle, we derive an exponential rate of convergence for solutions to the initial–boundary value problem for a class of quasilinear parabolic equations in one space dimension. We then apply the result to some models in population dynamics and image processing.


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*Keywords:* Quasilinear parabolic equations; Exponential rate of convergence; Maximum principle; Comparison principle; Aggregation in population dynamics; Perona–Malik model

## 1. Introduction

In this paper, we study large time behaviors of solutions to the initial–boundary value problem for a class of quasilinear advection–diffusion equations in one space dimension:

$$\begin{cases} \rho_t = (\sigma(\rho))_{xx} & \text{in } \Omega \times (0,\infty), \\ \rho = \rho_0 & \text{on } \Omega \times \{t=0\}, \\ \rho(0,t) = \rho(L,t) = 0 & \text{for } t \in (0,\infty). \end{cases} \tag{1.1}$$

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Here, $\Omega = (0, L) \subset \mathbb{R}$ is the spatial domain of a given length $L > 0$, $\sigma = \sigma(s) \in C^2(\mathbb{R})$ is the flux function, $\rho_0 = \rho_0(x)$ is a given initial datum, and $\rho = \rho(x, t)$ is a solution to the problem.

After the pioneering work of Zelenjak [18] and Matano [12], there have been many studies on the analysis of $\omega$-limit sets and large time behaviors of solutions to some semilinear or quasilinear parabolic problems in one space dimension; see, e.g., [2,17,5]. In particular, the work [17] considered a general quasilinear problem, which includes problem (1.1) as a special case, and proved the convergence of a global classical solution to a unique steady state in the space $C^1(\bar{\Omega})$ as $t \to \infty$ if the solution itself is assumed to be bounded in $C^1(\bar{\Omega})$ uniformly in $t \geq 0$.

In the present article, we show that a global classical solution to problem (1.1) converges uniformly to the steady state 0 as $t \to \infty$ at an exponential rate. The difference of our result from that in [17] is in two-fold. First, we do not assume the uniform boundedness of the solution to (1.1) in $C^1(\bar{\Omega})$. Second, we obtain specific exponential rates of convergence depending only on the initial datum $\rho_0$, the flux function $\sigma$ and the size $L$ of the spatial domain $\Omega$.

The main motivation of our result lies in its application to study large time behaviors of global *weak* solutions to some problems modeling aggregative movement in population dynamics [16, 1] and enhancement of a noisy picture in image processing [14]. However, proving the global existence of such solutions is not at all obvious since those problems are often *forward* and *backward parabolic* so that the standard methods of parabolic equations are not applicable. Actually, such existence results can be obtained through a combination of the result of this paper and the method of *convex integration* as one can see in a recent work of the author and Yan [10]. In this paper, we apply the main result to address large time behaviors of *classical* solutions to such problems for suitable initial data.

We now state the main result of the paper as follows.

**Theorem 1.1.** *Assume that $\sigma' > 0$ in $\mathbb{R}$. Let $\rho \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty))$ be a solution to problem (1.1), where $\rho_0 \in C(\bar{\Omega})$ is an initial datum satisfying the compatibility condition $\rho_0(0) = \rho_0(L) = 0$. Then for all $t \geq 0$, one has*

$$
\begin{aligned}
\|\rho(\cdot, t)\|_\infty \leq & \|\rho_0\|_\infty \frac{\max\left\{\frac{\|\rho_0\|_\infty \tilde{\theta}}{(1-\tau)\theta} + 1, m\right\} - e^{-\lambda L}}{\max\left\{\frac{\|\rho_0\|_\infty \tilde{\theta}}{(1-\tau)\theta} + 1, m\right\} - 1} \\
& \times \exp\left( - \frac{\tau\theta\lambda^2 e^{-\lambda L}}{\max\left\{\frac{\|\rho_0\|_\infty \tilde{\theta}}{(1-\tau)\theta} + 1, m\right\} - e^{-\lambda L}} t \right),
\end{aligned}
\tag{1.2}
$$

*where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}$, $0 < \tau < 1$, $\lambda > 0$, $m > 1$,*

$$
\theta := \min_{[-\|\rho_0\|_\infty \frac{m}{m-1}, \|\rho_0\|_\infty \frac{m}{m-1}]} \sigma' \text{ and } \tilde{\theta} := \max_{[-\|\rho_0\|_\infty \frac{m}{m-1}, \|\rho_0\|_\infty \frac{m}{m-1}]} |\sigma''|.
$$

The constants $\tau$, $\lambda$ and $m$ in the theorem can be chosen arbitrarily as described to fix the rate of convergence; once these numbers are fixed, it is clear that the rate of convergence depends only on $\rho_0$, $\sigma$ and $L$.

The rest of the paper is organized as follows. In Section 2, we derive appropriate maximum and comparison principles for qualitative behaviors of solutions to problem (1.1). Based on such a comparison principle, the proof of the main result, Theorem 1.1, is provided in Section 3.

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