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Dual exponential polynomials and linear differential equations

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Abstract

We study linear differential equations with exponential polynomial coefficients, where exactly one coefficient is of order greater than all the others. The main result shows that a nontrivial exponential polynomial solution of such an equation has a certain dual relationship with the maximum order coefficient. Several examples illustrate our results and exhibit possibilities that can occur.

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1. Introduction

The solutions of the linear differential equation

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = 0 \tag{1.1}$$

with entire coefficients $a_0(z), \ldots, a_{n-1}(z)$ are entire. To avoid ambiguity, we assume that $a_0(z) \not\equiv 0$. The following two classical theorems give general information concerning finite order solutions of (1.1).

Wittich's theorem. *[18, p. 6] The coefficients $a_0(z)$, ..., $a_{n-1}(z)$ of (1.1) are polynomials if and only if all solutions of (1.1) are of finite order.*

Frei's theorem. *[1, p. 207] Suppose that at least one coefficient in (1.1) is transcendental, and that $a_j(z)$ is the last transcendental coefficient, that is, the coefficients $a_{j+1}(z), \ldots, a_{n-1}(z)$, if applicable, are polynomials. Then (1.1) possesses at most j linearly independent solutions of finite order.*

The following example illustrates Frei's theorem for $n = 3$ and $j = 2$.

Example 1. The functions $f_1(z) = e^z + z$, $f_2(z) = e^z - 1$ and $f_3(z) = z + 1$ are solutions of

$$f''' + \left(z - 1 + e^{-z}\right) f'' - (z + 1)f' + f = 0,$$

and any two of them are linearly independent.

Our focus is on finite order solutions of (1.1), in particular on exponential polynomial solutions. An *exponential polynomial* $f(z)$ is a function of the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)}, \tag{1.2}$$

where $P_j(z)$, $Q_j(z)$ are polynomials for $1 \le j \le k$. Observe that a polynomial is a special case of an exponential polynomial.

The literature contains numerous examples of exponential polynomial solutions of equations of the form (1.1) with exponential polynomial coefficients such that at least one coefficient is transcendental. This naturally occurring situation is related to the long-standing open problem of Gol'dberg, Ostrovskiǐ and Petrenko, which asks whether finite order transcendental solutions of (1.1) are always of completely regular growth whenever the coefficients of (1.1) are of completely regular growth, see Section 2.

It can also be noted that any exponential polynomial is a solution of an equation of the form (1.1) with polynomial coefficients, see [17].

Motivated by these considerations, we consider the situation when f is an exponential polynomial solution of (1.1) where exactly one coefficient, say $a_\mu(z)$, is a transcendental exponential polynomial, and all the other coefficients are exponential polynomials of order less than the order of a_μ. In this case, we show that f and $a_\mu(z)$ have a certain *dual* relationship. Several examples are given to illustrate the main result.

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