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Global existence to the initial–boundary value problem for a system of nonlinear diffusion and wave equations

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Abstract

We prove the global existence of weak solution pair to the initial boundary value problem for a system of m-Laplacian type diffusion equation and nonlinear wave equation. The interaction of two equations is given through nonlinear source terms $f(u,v)$ and $g(u,v)$.

MSC: 35K92; 35L71; 35M13; 35M30

Keywords: Global existence; System; Nonlinear diffusion equation; Nonlinear wave equation

1. Introduction

In this paper we consider the initial–boundary value problem for a system of nonlinear diffusion and wave equations of the form:

$$\begin{cases} u_t - \operatorname{div}\{|\nabla u|^m \nabla u\} = f(u,v) \text{ in } \Omega \times (0,\infty), & \text{(a)} \\ v_{tt} - \Delta v = g(u,v) \text{ in } \Omega \times (0,\infty), & \text{(b)} \end{cases} \tag{1.1}$$

with the initial–boundary conditions

$$u(x,0) = u_0(x),\, v(x,0) = v_0(x),\, v_t(x,0) = v_1(x),\, x \in \Omega, \tag{1.2}$$

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and

$$u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0, \tag{1.3}$$

where $m \geq 0$, Ω is a bounded domain in R^N with a smooth, say, $C^{2,\alpha}$, $\alpha > 0$, class boundary $\partial\Omega$. When $f(u,v) = f(u)$ and $g(u,v) = g(v)$ the equations are separated mutually, and both of existence and nonexistence of global solutions are investigated by many authors. In particular, as a common method we know the potential well method due to Sattinger (see Sattinger [15] and M. Tsutsumi [18]). However the method does not seem to be applicable to the system of two equations like (1.1a)–(1.1b) except for a special case (cf. Alves and others [2]).

Both of diffusion and wave equations are very typical partial differential equations in Mathematical Physics, and it seems natural to imagine some phenomena caused by interactions between two equations (cf. Nishida [14], Li [6], Smoller [16], Kawashima [3], Kawashima, Matsumura and Nishihara [4] and Nishihara [13] etc.). In the present paper we consider a model of such a system where the interaction is given through nonlinear source terms.

We make the following hypotheses on $f(u,v)$ and $g(u,v)$.

Hyp. A. $f(u,v)$ and $g(u,v)$ are Lipschitz continuous (uniformly on each compact set) functions on R^2 and satisfy the following conditions:

(1)

$$|f(u,v)| \leq k_1|u|^\alpha|v|^\beta,\ |g(u,v)| \leq k_1|u|^p|v|^q$$

with $\alpha \geq 1$, $\beta \geq 0$, $p > 0$ and $q \geq 0$ for some $k_1 > 0$.

(2)

$$|f_u(u,v)| \leq k_2|u|^{\alpha-1}|v|^\beta,\ |f_v(u,v)| \leq k_2|u|^\alpha|v|^{\max\{\beta-1,0\}}\ (a.e.\ (u,v) \in R^2)$$

and

$$|g_u(u,v)| \leq k_2|u|^{p-1}|v|^q,\ |g_v(u,v)| \leq k_2|u|^p|v|^{\{q-1,0\}}\ (a.e.\ (u,v) \in R^2) \text{ for some } k_2 > 0.$$

A simple example of $g(u,v)$ satisfying the conditions above with $0 < p, q < 1$ is $g(u,v) = u(1+u^2)^{(p-1)/2}v(1+v^2)^{(q-1)/2}$. Under our Hyp. A we can expect that for small initial datum $u_0 \in L_r$, $r > 1$, the assumed solution $u(t)$ would have a smoothing effect near $t = 0$ and a certain decay property for $\|\nabla u(t)\|_{m+2}$ as $t \to \infty$. Further, using these properties we could prove the boundedness of the energy $E_v(t)$ of the assumed solution $v(t)$ of the second equation under appropriate conditions on α, β, p, q and r, where

$$E_v(t) \equiv \frac{1}{2}\left(\|v_t(t)\|^2 + \|\nabla v(t)\|^2\right).$$

These a priori estimates would imply the global existence of $(W_0^{1,m+2}, H_0^1)$ valued solution pair $(u(t), v(t))$. In this paper we show that such a conjecture in fact holds under some restrictions on the exponents. More precisely we show the existence of (weak) solution pair (u,v) such that

$$u(t) \in C_w([0,\infty); L^r) \cap L^\infty_{loc}((0,\infty) : W_0^{1,m+2}) \cap W^{1,2}_{loc}((0,\infty); L^2)$$

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