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Bifurcation of positive solutions to scalar reaction–diffusion equations with nonlinear boundary condition

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Abstract

The bifurcation of non-trivial steady state solutions of a scalar reaction–diffusion equation with nonlinear boundary conditions is considered using several new abstract bifurcation theorems. The existence and stability of positive steady state solutions are proved using a unified approach. The general results are applied to a Laplace equation with nonlinear boundary condition and bistable nonlinearity, and an elliptic equation with superlinear nonlinearity and sublinear boundary conditions.

MSC: 35B32; 35B35; 35K57; 35J61

Keywords: Reaction–diffusion equation; Nonlinear boundary condition; Bifurcation; Stability

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1. Introduction

Reaction–diffusion systems are considered as foremost important mathematical models for self-organized spatiotemporal pattern formation in morphogenesis, chemical reactions, ecology and various other scientific disciplines [4,19,31,40]. Typically one or several nonlinear parabolic equations describing the change of concentration of chemical or biological substances are defined in a spatial domain, which can be a cell or a tissue, a reactor or a Petri dish, or a habitat in the land or in the water depending on the context of the model. The spatial domain is often assumed to be an open bounded subset in $\mathbb{R}^N$ with $N = 1, 2$ or 3. For the wellposedness of the partial differential equation models, boundary conditions on the boundary of the spatial domain have to be defined so the equation is well-posed. Typical boundary conditions include Dirichlet one for which the value of state variables on the boundary are specified, or Neumann one for which the flux of state variables across the boundary are known. The homogeneous Neumann boundary condition assumes a zero flux on the boundary, and it represents insulation or a closed system. All these boundary conditions are linear equations of the functions and their normal derivatives on the boundary.

In many other situations, the chemical reactions or the biological bondings occur in a narrow layer near the boundary or on the boundary surface (cell membrane), and the nonlinear reaction on the boundary makes a nonlinear boundary condition. For example, a highly exothermic reaction can take place in a thin layer around a boundary [20], and a Bcd gradient formation are generated through a source on the boundary [15,19]. Such reaction–diffusion models are usually consisted of a linear diffusion equation in the spatial domain and a nonlinear reaction on the boundary [2,20,37], and they may represent some pattern formation mechanisms different from the classical ones derived from interior reaction and fixed boundary conditions.

Several aspects of reaction–diffusion models with nonlinear boundary conditions have been considered. The wellposedness and asymptotical behavior of solutions were studied in [1,2,8,30,37]; the blowup of solutions were characterized in [17,21,25,48,49]; and the boundary layer solutions were constructed in [3,5,11,12]. The existence, uniqueness and stability of steady state solutions have also been studied via bifurcation method and other related methods in several special classes of problems [6,7,13,27,28,33–35,41,42,46,47]. On the other hand, a general bifurcation theorem was recently established in [39], which provides a more direct approach to the bifurcation under nonlinear boundary conditions.

In this paper, we provide a unified approach for the bifurcation of non-trivial steady state solutions of a scalar reaction–diffusion equation with nonlinear boundary conditions. More precisely we consider a scalar parabolic equation with a nonlinear boundary condition in form

$$\begin{cases} u_t = \Delta u + \lambda s(x) f(u), & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) g(u), & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_*(x), & x \in \Omega, \end{cases} \tag{1.1}$$

and the steady state solutions of (1.1) satisfy

$$\begin{cases} -\Delta u = \lambda s(x) f(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \lambda r(x) g(u), & x \in \partial\Omega. \end{cases} \tag{1.2}$$

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