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On the number of polynomial solutions of Bernoulli and Abel polynomial differential equations

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Abstract

In this paper we determine the maximum number of polynomial solutions of Bernoulli differential equations and of some integrable polynomial Abel differential equations. As far as we know, the tools used to prove our results have not been utilized before for studying this type of questions. We show that the addressed problems can be reduced to know the number of polynomial solutions of a related polynomial equation of arbitrary degree. Then we approach to these equations either applying several tools developed to study extended Fermat problems for polynomial equations, or reducing the question to the computation of the genus of some associated planar algebraic curves.

MSC: primary: 34A05; secondary: 11D41, 11R09, 34A34, 35C11, 37C10

Keywords: Abel equation; Ricatti equation; Bernouilli equation; Generalized Fermat theorem for polynomials; Polynomial solution

1. Introduction

In this work we investigate the number of polynomial solutions of some differential equations of type

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_{n-1}(t)\,x^{n-1} + \cdots + p_1(t)\,x + p_0(t) \tag{1}$$

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with q and p_i polynomials in real or complex coefficients for $i = 0, 1, 2, \dots, n$, and $p_n(t) \not\equiv 0$. More specifically, we consider the real or complex Bernoulli equation ($p_{n-1} = p_{n-2} = \cdots = p_1 = 0$) and some special real Abel equations ($n = 3$) that will be fixed below.

There are several previous works asking for polynomial solutions of equation (1) for some values of n.

When $n = 2$, equation (1) is the well-known polynomial Riccati equation. In 1936, Rainville proved the existence of one or two polynomial solutions when $q(t) = 1$, see [19]. After, in the papers [7,8] the authors presented some criteria determining the degree of polynomial solutions of $q(t)\,\dot{x} = p_2(t)\,x^2 + p_1(t)\,x + p_0(t)$ and show examples of these equations with 4 or 5 polynomial solutions. For them, in [10] the authors gave a complete answer: polynomial Riccati equations have at most $N + 1$ (resp. 2) polynomial solutions when $N \geq 1$ (resp. $N = 0$), where N is the maximum degree of $q(t)$, $p_0(t)$, $p_1(t)$, $p_2(t)$; moreover, there are equations of this type having any number of polynomial solutions smaller than or equal to these upper bounds.

Also in [2–4] the degrees of the polynomial solutions of (1) are studied. In this setting in [13] it is shown that the degree of the polynomial solutions of (1) has to belong to a particular set of integers depending on the degrees of the coefficients. Finally, in [11] it is proved that equation (1) with $q = 1$ has at most n polynomial solutions and that this bound is sharp.

Notice that the question we are interested in is also reminiscent of a similar one proposed by Poincaré about the number and degree of the algebraic solutions of planar autonomous polynomial differential systems in terms of their degrees.

Our first result solves completely the problem for Bernoulli equations. It is not difficult to prove that linear equations have 0, 1 or all its solutions being polynomials. For instance the equation (2) with $n = 0$, $\dot{x} = t$, has the solutions $x = t^2/2 + c$, $c \in \mathbb{C}$. As we have already explained, the case $n = 2$, is solved in [10]. We include it in next theorem for the sake of completeness.

Theorem A. *Consider Bernoulli equations*

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_1(t)\,x, \tag{2}$$

with $q, p_n, p_1 \in \mathbb{C}[t]$ *and* $p_n(t) \not\equiv 0$. *Then:*

- *(i) For* $n = 2$, *equation* (2) *has at most* $N + 1$ *(resp. 2) polynomial solutions, where* $N \geq 1$ *(resp.* $N = 0$*) is the maximum degree of* q, p_2, p_1, *and these upper bounds are sharp. Moreover, when* $q, p_2, p_1 \in \mathbb{R}[t]$ *these upper bounds are reached with real polynomial solutions.*
- *(ii) For* $n = 3$, *equation* (2) *has at most seven polynomial solutions and this upper bound is sharp. Moreover, when* $q, p_3, p_1 \in \mathbb{R}[t]$ *this upper bound is reached with seven polynomial solutions belonging to* $\mathbb{R}[t]$.
- *(iii) For* $n \geq 4$, *equation* (2) *has at most* $2n - 1$ *polynomial solutions and this upper bound is sharp. Moreover, when* $q, p_n, p_1 \in \mathbb{R}[t]$ *it has at most three real polynomial solutions when n is even while it has at most five real polynomial solutions when n is odd, and both upper bounds are sharp.*

Notice also, that in general, given $n + 1$ arbitrary polynomials $x_0, x_2, \dots, x_n$ there exists always an equation of the form (1) having these solutions as particular solutions. To get this differential equation it suffices to plug them in the equation (1) with $q = 1$ and solve the linear system with $n+1$ unknowns $p_n, p_{n-1}, \dots, p_0$. Solving it we obtain a rational differential equation. Multiplying this equation by the least common multiple of all the denominators of the p_j, we obtain

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