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New type of solutions to a slightly subcritical Hénon type problem on general domains

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Abstract

We consider the following slightly subcritical problem

$$(\wp_\varepsilon) \qquad \begin{cases} -\Delta u = \beta(x)|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $3 \leq n \leq 6$, $p := \frac{n+2}{n-2}$ is the Sobolev critical exponent, ε is a small positive parameter and $\beta \in C^2(\overline{\Omega})$ is a positive function. We assume that there exists a nondegenerate critical point $\xi_* \in \partial\Omega$ of the restriction of β to the boundary $\partial\Omega$ such that

$$\nabla(\beta(\xi_*)^{\frac{-2}{p-1}}) \cdot \eta(\xi_*) > 0,$$

where η denotes the inner normal unit vector on $\partial\Omega$. Given any integer $k \geq 1$, we show that for $\varepsilon > 0$ small enough problem $(\wp_\varepsilon)$ has a positive solution, which is a sum of k bubbles which accumulate at ξ_* as ε tends

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to zero. We also prove the existence of a sign changing solution whose shape resembles a sum of a positive bubble and a negative bubble near the point ξ_*.

MSC: 35J60; 35J20; 35J25

Keywords: Hénon problem; Critical exponent; Blowing up solutions

1. Introduction and statement of main results

We consider the nonautonomous almost critical problem

$$(\wp_\varepsilon) \qquad \begin{cases} -\Delta u = \beta(x)|u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $3 \le n \le 6$, $p := \frac{n+2}{n-2}$ is the Sobolev critical exponent, ε is a small positive parameter and the function $\beta \in C^2(\overline{\Omega})$ is positive.

Since problem $(\wp_\varepsilon)$ is subcritical, standard variational methods yields the existence of an infinite number of sign changing solutions and at least one positive solution, see [2]. Unfortunately, the variational approach gives very little information about the behaviour of these solutions.

A special case of problem $(\wp_\varepsilon)$ is the following

$$(\wp_\varepsilon^1) \qquad \begin{cases} -\Delta u = |u|^{p-1-\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has been extensively studied in the last decades and many works has been devoted to study existence and asymptotic behaviour of solutions. We refer to the pioneering work by Bahri–Li–Rey in [3], where they proved that positive solutions to problem $(\wp_\varepsilon^1)$ either converge to a positive solution of the critical problem $(\wp_0^1)$ or blow up at a finite number of points in Ω as ε goes to zero. More precisely, if (u_ε) is a bounded sequence in $H_0^1(\Omega)$ of positive solutions to $(\wp_\varepsilon^1)$, then (up to a subsequence) we have

$$u_\varepsilon = u_0 + \sum_{i=1}^{k} \alpha_i^\varepsilon PU_{\lambda_i^\varepsilon, \xi_i^\varepsilon} + v_\varepsilon$$

where u_0 is a nonnegative solution to $(\wp_0^1)$, $k \in \mathbb{N}$, v_ε goes to zero in $H_0^1(\Omega)$. Here either $k = 0$ or $u_0 = 0$ and the function $PU_{\delta,\xi}$ is the orthogonal projection onto $H_0^1(\Omega)$ of the "bubble" given by

$$U_{\delta,\xi}(x) := (n(n-2))^{\frac{n-2}{4}} \frac{\lambda^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}},$$

with $\delta > 0$ and $\xi \in \mathbb{R}^n$. This family of functions represent all solutions to