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A note on local integrability of differential systems

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Abstract

For an n-dimensional local analytic differential system $\dot{x} = Ax + f(x)$ with $f(x) = O(|x|^2)$, the Poincaré nonintegrability theorem states that if the eigenvalues of A are not resonant, the system does not have an analytic or a formal first integral in a neighborhood of the origin. This result was extended in 2003 to the case when A admits one zero eigenvalue and the other are non-resonant: for $n = 2$ the system has an analytic first integral at the origin if and only if the origin is a non-isolated singular point; for $n > 2$ the system has a formal first integral at the origin if and only if the origin is not an isolated singular point. However, the question of *whether the system has an analytic first integral at the origin provided that the origin is not an isolated singular point* remains open.

MSC: 34A34; 34C20; 34C45; 37J30; 37K10

Keywords: Analytic differential systems; Non-isolated singular point; Local integrability; Invariant manifold

1. Introduction and statement of the main results

For the local analytic differential system

$$\dot{x} = Ax + f(x), \qquad x \in \mathbb{R}^n, \tag{1}$$

with $f(x) = O^*(\|x\|^2) \in C^\omega(\mathbb{R}^n, 0)$, the study of the theory of local integrability or of the existence of first integrals at the origin can be traced back to Poincaré [16]. Since then, the theory of

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local integrability has been greatly developed, see for example [1,2,4–6,8,15,17–22]. Hereafter, $O^*(\|x\|^2)$ denotes a function (or a vector-valued function) without constant and linear terms in its Taylor expansion, and $C^\omega(\mathbb{R}^n, 0)$ denotes the set of analytic functions defined in a neighborhood of the origin. Note that $x = 0$ is a singular point of system (1), and that after an invertible linear change of coordinates we can always transform system (1) to a system with its linear part matrix in Jordan normal form. So in what follows, we assume without loss of generality that A is in Jordan normal form.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of the matrix A. Set

$$\mathcal{M}_\lambda := \left\{ m \in \mathbb{Z}_+^n | \ \langle m, \lambda \rangle = 0, \ |m| \geq 1 \right\},$$

where $\mathbb{Z}_+$ is the set of nonnegative integers, $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in $\mathbb{C}^n$ and $|m| = m_1 + \ldots + m_n$ for $m = (m_1, \ldots, m_n)$. If $\mathcal{M}_\lambda = \emptyset$, we say λ is *non-resonant*. If $\mathcal{M}_\lambda \neq \emptyset$, each element of $\mathcal{M}_\lambda$ is called a *resonant lattice*.

Poincaré [16] proved the next result.

Theorem A. *If system (1) is analytic, and the eigenvalues λ of A is non-resonant, then the system has neither analytic nor formal first integrals.*

For a proof of the Poincaré's result, see example [8] or [19].

Recall that a formal first integral is a formal series $H(x)$ which satisfies $\langle \nabla H(x), Ax + f(x) \rangle \equiv 0$ in $(\mathbb{R}^n, 0)$, where ∇H is the gradient of H and the partial derivative of H is taken over all homogeneous terms in the sum of H, that is, if $H(x) = \sum_{\ell=1}^{\infty} H_\ell(x)$ with the H_ℓ's homogeneous polynomials of degree ℓ then $\nabla H = \sum_{\ell=1}^{\infty} \nabla H_\ell(x)$.

When the n-tuple of eigenvalues λ are resonant, i.e. $\mathcal{M}_\lambda \neq \emptyset$, there are certain known results which provide necessary conditions ensuring the existence and number of functionally independent local analytic or formal first integrals of system (1). For more details, see [2,4,6,15,18]. About the equivalent characterization of analytic integrability via normal form, there are also some known results on the existence of analytic normalization of analytically integrable differential systems to their Poincaré–Dulac normal forms. For more details, see [11,12,21–24]. As our knowledge, there are very few general results providing necessary and sufficient conditions for the existence of analytic or formal first integrals defined in a neighborhood of the origin.

Li *et al* [14] in 2003 studied the existence of local first integrals at the origin in case when one of the eigenvalues vanishes and the other are non-resonant, that is

$$\lambda_1 = 0 \ \text{ and } \ \sum_{j=2}^{n} m_j \lambda_j \neq 0 \ \text{ for } m_j \in \mathbb{Z}_+ \text{ and } \sum_{j=2}^{n} m_j \geq 1. \tag{2}$$

Their results can be stated as follows.

Theorem B. *Assume that the differential system (1) is analytic and the conditions (2) hold.*

(*a*) *For $n > 2$, system (1) has a formal first integral in a neighborhood of $x = 0$ if and only if the singular point $x = 0$ is not isolated. In particular, if the singular point $x = 0$ is isolated, system (1) has no analytic first integrals in a neighborhood of $x = 0$.*