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Convergence to equilibrium of solutions to a nonautonomous semilinear viscoelastic equation with finite or infinite memory

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Abstract

In this paper we consider the nonautonomous semilinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^\tau k(s)\Delta u(t-s)ds + f(x,u) = g,\ \ \tau \in \{t, \infty\},$$

in $\mathbb{R}^+ \times \Omega$, with Dirichlet boundary conditions and finite ($\tau = t$) or infinite ($\tau = \infty$) memory. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and the nonlinearity $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is analytic in the second variable, uniformly with respect to the first one. For this equation, we derive an appropriate Lyapunov function and we use the Łojasiewicz–Simon inequality to show that the dissipation given by the memory term is strong enough to prove the convergence to a steady state for any global bounded solution. In addition, we discuss the rate of convergence to equilibrium which is polynomial or exponential, depending on the Łojasiewicz exponent and the decay of the time-dependent right-hand side g.

MSC: 28C15; 46E05; 28C99

Keywords: Evolutionary integral equation; Semilinear; Stabilization; Łojasiewicz–Simon inequality

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1. Introduction and assumptions

The aim of this paper is the study of the convergence and decay rate to a steady state of global bounded solutions of the following nonautonomous semilinear viscoelastic equation with finite memory

$$\begin{cases} u_{tt} - \Delta u + \int_0^t k(s)\Delta u(t-s)ds + f(x,u) = g \text{ in } \mathbb{R}^+ \times \Omega, \\ u = 0 \text{ on } \mathbb{R}^+ \times \Gamma, \\ u(0) = u_0 \text{ in } \Omega, \\ u_t(0) = u_1 \text{ in } \Omega, \end{cases} \tag{1}$$

and with infinite memory

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty k(s)\Delta u(t-s)ds + f(x,u) = g \text{ in } \mathbb{R}^+ \times \Omega, \\ u = 0 \text{ on } \mathbb{R}^+ \times \Gamma, \\ u(-t) = u^0(t) \text{ for } t \geq 0, \\ u_t(0) = u_1 \text{ in } \Omega, \end{cases} \tag{2}$$

where $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ is a bounded open set with smooth boundary Γ and the functions $u_0,\ u_1 : \Omega \to \mathbb{R}$ and $u^0 : \mathbb{R} \times \Omega \to \mathbb{R}$ are given initial data. The relaxation function k, the nonlinearity f and the forcing term g will be specified later. These problems model some phenomena in viscoelasticity. Here, we understand $\int k(s)\Delta u(t-s)ds$, $f(x,u)$, and $g(t)$ to be the viscoelastic term, the source term, and the exterior forcing term, respectively; we refer the reader to [12,15,31] for a discussion on how these models arise. See also [16–18] and the references therein for more details concerning the physical phenomena which are modelled by differential equations with memory.

In the present work we study the asymptotic behaviour of global bounded weak solutions for problems (1) and (2) as $t \to \infty$. The proof is based on the construction of an appropriate new Lyapunov functional, compactness properties, and on the Łojasiewicz–Simon inequality. In particular, when the kernel $k(t)$ decays exponentially, we show that, if g tends to 0 sufficiently fast at infinity, any bounded solution of (1) and (2) having a relatively compact range in the energy space converges to a single steady state. We also give a condition implying existence of bounded global solutions. Finally, we show that the decay rate to equilibrium is either exponential or polynomial.

The asymptotic behaviour as time goes to infinity of global solutions of problems (1) and (2) has been studied by many mathematicians, and related results concerning existence, convergence to equilibrium, and blow-up of solutions have been recently established.

For the autonomous linear case (i.e. $f = g = 0$), a number of papers have appeared covering a large class of kernels k, which guarantee stability and show the relation between the decay rate of k and the asymptotic behaviour of solutions of the considered problem. Examples of such results can be found, for instance, in [4,13,16,19,26–29] and the references therein.

An autonomous semilinear initial value problem related to Eq. (1) is considered in [8], namely

$$u_{tt} - \Delta u + \int_0^t k(s)\Delta u(t-s)ds + |u|^\rho u = 0 \text{ in } \mathbb{R}^+ \times \Omega, \tag{3}$$

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