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Existence of self-shrinkers to the degree-one curvature flow with a rotationally symmetric conical end

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Abstract

Given a smooth, symmetric, homogeneous of degree one function $f(\lambda_1, \cdots, \lambda_n)$ satisfying $\partial_i f > 0$ for all $i = 1, \cdots, n$, and a rotationally symmetric cone $\mathcal{C}$ in $\mathbb{R}^{n+1}$, we show that there is a f self-shrinker (i.e. a hypersurface Σ in $\mathbb{R}^{n+1}$ which satisfies $f(\kappa_1, \cdots, \kappa_n) + \frac{1}{2} X \cdot N = 0$, where X is the position vector, N is the unit normal vector, and $\kappa_1, \cdots, \kappa_n$ are principal curvatures of Σ) that is asymptotic to $\mathcal{C}$ at infinity.

1. Introduction

Let $\mathcal{C}$ be a rotationally symmetric cone in $\mathbb{R}^{n+1}$, say

$$\mathcal{C} = \left\{ (\sigma s\, \nu,\, s) \,\middle|\, \nu \in \mathbf{S}^{n-1},\, s \in \mathbb{R}_+ \right\}$$

where $\sigma > 0$ is a constant. Let $\boldsymbol{\Sigma}$ be a properly embedded hypersurface in $\mathbb{R}^{n+1}$. Then $\boldsymbol{\Sigma}$ is called a self-shrinker to the mean curvature flow (MCF: motions of hypersurfaces whose normal velocity are given by the mean curvature vector) which is C^k asymptotic to the cone $\mathcal{C}$ at infinity provided that

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$$H + \frac{1}{2} X \cdot N = 0$$

$$\varrho \boldsymbol{\Sigma} \xrightarrow{C^k_{\rm loc}} \mathcal{C} \quad \text{as } \varrho \searrow 0$$

where X is the position vector, N is the unit normal vector and H is the mean curvature of $\boldsymbol{\Sigma}$. Note that the rescaled family of hypersurfaces $\left\{\boldsymbol{\Sigma}_t = \sqrt{-t}\, \boldsymbol{\Sigma}\right\}_{-1 \le t < 0}$ forms a MCF starting from $\boldsymbol{\Sigma}$ (when $t = -1$) and converging locally C^k to $\mathcal{C}$ as $t \nearrow 0$.

In the case when $\boldsymbol{\Sigma}$ is rotationally symmetric, one can parametrize it by

$$X(\nu, s) = (\mathtt{r}(s)\, \nu,\, s) \quad \text{for } \nu \in \mathbf{S}^{n-1},\ s \in (c_1, c_2)$$

for some constants $0 \le c_1 < c_2 \le \infty$. We may orient it by the unit-normal

$$N = \frac{(-\nu,\, \partial_s \mathtt{r})}{\sqrt{1 + (\partial_s \mathtt{r})^2}} \tag{1.1}$$

At each point $X \in \boldsymbol{\Sigma}$, choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for $T_X \boldsymbol{\Sigma}$ so that

$$e_n = \frac{\partial_s X}{|\partial_s X|} = \frac{(\partial_s \mathtt{r}\, \nu,\, 1)}{\sqrt{1 + (\partial_s \mathtt{r})^2}}$$

then $\{e_1, \cdots, e_n\}$ forms a set of principal vectors of $\boldsymbol{\Sigma}$ at X with principal curvatures

$$\kappa_1 = \cdots = \kappa_{n-1} = \frac{1}{\mathtt{r}\sqrt{1 + (\partial_s \mathtt{r})^2}}, \quad \kappa_n = \frac{-\partial_s^2 \mathtt{r}}{\left(1 + (\partial_s \mathtt{r})^2\right)^{\frac{3}{2}}} \tag{1.2}$$

As a result, $\boldsymbol{\Sigma}$ is a rotationally symmetric self-shrinker to the MCF if and only if

$$\left(\frac{n-1}{\mathtt{r}} - \frac{\partial_s^2 \mathtt{r}}{1 + |\partial_s \mathtt{r}|^2}\right) + \frac{1}{2}\left(s\, \partial_s \mathtt{r} - \mathtt{r}\right) = 0 \tag{1.3}$$

Kleene and Moller showed in [3] that there exists a unique rotationally symmetric self-shrinker

$$\boldsymbol{\Sigma} :\ X(\nu, s) = (\mathtt{r}(s)\nu,\, s), \quad \nu \in \mathbf{S}^{n-1},\ s \in [R, \infty)$$

where the radius function $\mathtt{r}(s)$ satisfies (1.3) and

$$s \left| \mathtt{r}(s) - \sigma s \right| \le \frac{2(n-1)}{\sigma}, \quad s^2 \left| \partial_s \mathtt{r} - \sigma \right| \le \frac{2(n-1)}{\sigma}$$

The key step is to analyze the following representation formula for (1.3):

$$\mathtt{r}(s) = \sigma s +$$

$$s \int_s^\infty \frac{1}{x^2} \left\{ \int_x^\infty \xi \exp\left(-\frac{1}{2} \int_x^\xi \boldsymbol{\tau} \left(1 + (\partial_s \mathtt{r}(\boldsymbol{\tau}))^2\right) d\boldsymbol{\tau} \right) \left[\frac{n-1}{\mathtt{r}(\xi)} \left(1 + (\partial_s \mathtt{r}(\xi))^2\right) \right] d\xi \right\} dx$$