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New classes of non-convolution integral equations arising from Lie symmetry analysis of hyperbolic PDEs

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Abstract

In this paper we consider some new classes of integral equations that arise from Lie symmetry analysis. Specifically, we consider the task of obtaining solutions of a Cauchy problem for some classes of second order hyperbolic partial differential equations. Our analysis leads to new integral equations of non-convolution type, which can be solved by classical methods. We derive solutions of these integral equations, which in turn lead to solutions of the associated Cauchy problems.

Keywords: Hyperbolic PDEs; Integral equations; Fundamental solutions; Cauchy problems; Harmonic analysis

1. Introduction

The theory of continuous symmetry groups of systems of partial differential equations was developed by Lie in the last decades of the nineteenth century. A symmetry is a transformation which maps solutions to other solutions. Lie developed a method for systematically computing all continuous symmetries of a given system of differential equations. Excellent modern accounts may be found in the books by Olver [17] and those of Bluman and his coauthors, such as the volume [2].

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Symmetries are powerful tools and they allow us to solve a wide variety of problems. For example, we may compute fundamental solutions of parabolic equations from a knowledge of their symmetries. See the papers [10], [11], [6], [9] for details, together with many examples and applications. Boundary value problems may also be solved by the method of group invariant solutions. The aforementioned book [2] contains an extensive discussion of this topic. Symmetries are also essential to the study of conservation laws. A chapter of [17] is devoted to this.

A new way of applying Lie symmetries was introduced in [6] and developed extensively in [7]. The idea is to construct an integral operator that maps test functions to solutions of the PDE by integration against a symmetry solution. The purpose of this paper is to show how the method can lead to new classes of non-convolution integral equations, which may be solved by a combination of classical techniques.

We first provide an example to illustrate how the method works. We then turn to a class of second order linear hyperbolic equations and derive some new integral equations which arise in the solution of Cauchy problems associated to these equations. We then solve these equations using a novel combination of integral transform methods.

The outline of the paper is as follows. In Section 2 we introduce the method that forms the basis of our analysis. In Section 3 we determine the symmetries of a class of hyperbolic PDEs of the form $u_{tt} = u_{xx} + f(x)u$. We show that there are nontrivial symmetries when f satisfies one of three families of Riccati equations. In Section 4 we study the equation $u_{tt} = y^2u_{yy} + yu_y + y^2u$ (which is equivalent to $u_{tt} = u_{xx} + e^{2x}u$, with $y = e^x$) and show how the symmetries lead to a solution of the Cauchy problem for this equation via new non-convolution integral equations, which we are able so solve. See Theorems 4.2 and 4.3. In Section 5 we set up the integrals for the Cauchy problem for the more general equation $u_{tt} = u_{xx} + (Ae^x + Be^{-x})^{-2}u$. In Section 6 we solve the integral equation in the case $A = B = 1/2$ and obtain a solution of the problem $u_{tt} = u_{xx} + (\text{sech}^2 x)u$, $x > 0$, subject to $u(x, 0) = f(x)$, $u_t(x, 0) = 0$. (See Theorem 6.6.) A brief conclusion follows. We believe that the methods of this paper may prove to be of use in the study of various types of linear PDEs.

2. Introduction to the method

We illustrate the approach by solving the problem

$$u_{tt} = u_{xx} - \frac{A}{x^2}u,\ \ u(x,0) = f(x), u_t(x,0) = g(x),\ \ x > 0, t \geq 0.$$

This problem was solved using different symmetries in [7]. Here we use a simpler approach. We suppose that $f, g \in \mathcal{S}(\mathbb{R}_+)$ the Schwartz space, consisting of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power, i.e. Schwartz functions are rapidly decreasing. It is known that the Fourier Transform and Mellin transforms are automorphisms of the Schwartz space, see [22]. This space is a topological vector space of functions φ such that $\varphi \in C^\infty(\mathbb{R}_+)$ and $x^\alpha \varphi^{(\beta)}(x) \to 0$, $x \to \infty$, $\alpha, \beta \in \mathbb{N}_0$. We note here that throughout the paper $L_1(\sigma)$ will denote the usual Lebesgue space of measurable functions f such that $\int_\sigma |f(s)|ds < \infty, \sigma = \{s \in \mathbb{C} : s = \frac{1}{2} + i\tau, \tau \in \mathbb{R}\}$.

It is elementary that the scaling transformation $u(x,t) \to u(\lambda x, \lambda t)$ is a symmetry. The idea is to introduce a new solution by setting