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Invariant regions for systems of lattice reaction–diffusion equations

Antonín Slavík

Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic

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Abstract

In this paper, we study systems of lattice differential equations of reaction–diffusion type. First, we establish some basic properties such as the local existence and global uniqueness of bounded solutions. Then we proceed to our main goal, which is the study of invariant regions. Our main result can be interpreted as an analogue of the weak maximum principle for systems of lattice differential equations. It is inspired by existing results for parabolic differential equations, but its proof is different and relies on the Euler approximations of solutions to lattice differential equations. As a corollary, we obtain a global existence theorem for nonlinear systems of lattice reaction–diffusion equations. The results are illustrated on examples from population dynamics.

Keywords: Lattice differential equation; Reaction–diffusion equation; Invariant region; Maximum principle; Existence and uniqueness

1. Introduction

The most studied example of a lattice differential equation has the form

$$\frac{\partial u}{\partial t}(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \; t \geq 0, \quad (1.1)$$

E-mail address: slavik@karlin.mff.cuni.cz.

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where $u : \mathbb{Z} \times [0, \infty) \to \mathbb{R}$ is the unknown function. This equation is obtained from the classical one-dimensional reaction–diffusion equation

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t),x,t), \quad x \in \mathbb{R}, \quad t \in [0,\infty), \tag{1.2}$$

by discretizing the space variable. For some applications in biology, chemistry, kinematics or population dynamics, the semidiscrete equation seems to be more appropriate than the classical reaction–diffusion equation (see, e.g., [2,11,14,21,22]).

For various choices of the reaction function f, numerous authors have studied the properties of solutions to Eq. (1.1), such as the asymptotic behavior [4,35,36], existence of traveling wave solutions [8,11,24,40,41] or pattern formation [6–8]. On the other hand, the recent papers [30, 31] have focused on well-posedness results and maximum principles for Eq. (1.1) with a general reaction function f. Let us mention that the maximum principles are important for the study of traveling wave solutions (cf. [24,39]).

Systems of two or more lattice differential equations were also considered by numerous authors. The motivation for the study of such systems often comes from population dynamics – see, e.g., [5,16–19,23] and the references there. Again, most papers focus on equations of reaction–diffusion type with specific choices of the reaction function. A fairly general class of linear lattice systems with continuous, discrete or mixed time was studied in [28].

The present paper is devoted to general systems of nonlinear lattice differential equations of the form

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= A(x,t)u(x+1,t) + B(x,t)u(x,t) + C(x,t)u(x-1,t)\\ &\quad + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \geq 0,\end{aligned} \tag{1.3}$$

where u takes values in $\mathbb{R}^m$ and A, B, C are matrix-valued functions. Obviously, Eq. (1.1) represents a special case of Eq. (1.3) with $m = 1$, $A(x,t) = C(x,t) = k$ and $B(x,t) = -2k$.

In Section 2, we present some basic results on the existence and uniqueness of solutions to nonlinear systems of lattice equations. We focus on initial-values problems with bounded initial conditions. Such problems generally have infinitely many solutions (see, e.g., [29, Section 3]); to get uniqueness, we restrict ourselves to the class of bounded solutions. As explained in [14], the space of bounded sequences is a quite natural choice for the study of diffusion-type lattice differential equations.

The core of the paper is in Section 3, where we study invariant regions for systems of the form (1.3). The invariance results can be interpreted as a generalization of the weak maximum principle: In the scalar case (1.1), the weak maximum principle says that under suitable assumptions on the reaction function f, the values of the solution always remain in the interval determined by the infimum and supremum of the initial values. Thus, the interval is an invariant region for the given equation. In the higher-dimensional setting, the interval is replaced by a closed convex set S, and the problem is to find sufficient conditions guaranteeing that S is an invariant region, i.e., that solutions with initial values in S never leave this set. The key assumption is that the vector field f points inward S or is tangent to the boundary at all boundary points of S. This condition is well known from the invariance results for classical parabolic equations; see [1,9,25,34,38]. The proofs of these classical results are fairly straightforward for bounded spatial domains, while the treatment of unbounded domains is more difficult.

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