ARTICLE IN PRESS YJDEQ:8955

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

Dynamics of degenerate quasilinear reaction diffusion systems with nonnegative initial functions

C.V. Pao [a], W.H. Ruan [b,*]

[a] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, United States*
[b] *Department of Mathematics, Statistics and Computer Science, Purdue University Northwest, Hammond, IN 46323-2094, United States*

Received 11 April 2017

Abstract

This paper is concerned with a system of quasilinear reaction–diffusion equations with density dependent diffusion coefficients and mixed quasimonotone reaction functions. The equations are allowed to be degenerate and the boundary conditions are of the nonlinear type. The main goals are to prove the existence and uniqueness of the weak solution between a pair of coupled upper and lower solutions; show that the weak solution evolves into the classical solution, and analyze the asymptotic behavior of the solution using quasi-solutions of the steady-state system. The general results are applied to a degenerate Lotka–Volterra competition model. Conditions are given for the solution to exist globally, to evolve into the classical solution, and to be attracted into a sector formed by quasi-solutions of the elliptic system. Especially for the Neumann problem we give a simple condition for the solution to converge to a unique constant steady-state solution which is a global attractor.

MSC: primary 35K50, 35J55 ; secondary 35K65, 35K57

Keywords: Quasilinear reaction diffusion systems; Upper and lower solutions; Weak and classical solutions; Asymptotic behavior; Lotka–Volterra models

* Corresponding author.
E-mail addresses: cvpao@ncsu.edu (C.V. Pao), wruan@pnw.edu (W.H. Ruan).

http://dx.doi.org/10.1016/j.jde.2017.08.024
0022-0396/

Please cite this article in press as: C.V. Pao, W.H. Ruan, Dynamics of degenerate quasilinear reaction diffusion systems with nonnegative initial functions, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.08.024

1. Introduction

One of the most important concerns in the study of reaction diffusion systems is the various properties of the solution such as the existence-uniqueness of a classical or a weak solution, regularity property of the weak solution, and the asymptotic behavior of the solution. This kind of problem has been studied extensively in the literature but mostly are for semilinear reaction diffusion systems where the diffusion coefficients are independent of the density functions (cf. [23]). In recent years, attention has been given to quasilinear reaction–diffusion systems where the diffusion coefficients are density dependent and the reaction functions are mixed quasimonotone (cf. [2,4,9,10,25–28,30,31,33,34]). This consideration is directly applicable to many physical, biological and ecological problems, especially in relation to porous medium type of reaction diffusion systems. One of these problems is the Lotka–Volterra competition system which is given in the form

$$\begin{aligned} &\partial u_i/\partial t - d_i \Delta u_i^{m_i} = u_i \left(a_i - b_{ii} u_i - \textstyle\sum_{j\neq i}^{N} b_{ij} u_j \right) && (t>0, \quad x\in\Omega), \\ &B_i[u_i] \equiv \partial u_i/\partial \nu + \beta_i(x) u_i = h_i(x) && (t>0, \quad x\in\partial\Omega), \\ &u_i(0,x) = \psi_i(x) && (x\in\Omega), \quad i=1,\dots,N \end{aligned} \tag{1.0}$$

where d_i, m_i and a_i are positive constants with $m_i \geq 1$, b_{ij} are nonnegative constants, $\beta_i(x)$ and $h_i(x)$ are nonnegative smooth functions on $\partial\Omega$, Ω is a bounded domain in $\mathbb{R}^p$ with boundary $\partial\Omega$, and $\partial/\partial\nu$ denotes the outward normal derivative of u_i on $\partial\Omega$. The dynamics of the above system for the semilinear case $m_i = 1$ has been extensively investigated in both mathematical and biological literature (cf. [5,6,8,11,12,15,18,20–22,24]). It has also been studied recently in [25] for the quasilinear case $m_i > 1$ using the method of upper and lower solutions. It is shown in [25] that if problem (1.0) has a pair of coupled upper and lower solutions $\tilde{\mathbf{u}}(t,x)$ and $\hat{\mathbf{u}}(t,x)$ with $\tilde{\mathbf{u}}(t,x) \geq \hat{\mathbf{u}}(t,x) \geq \mathbf{0}$ (see Definition 2.1 in Section 2), then it has a unique classical solution $\mathbf{u}(t,x)$ between $\tilde{\mathbf{u}}(t,x)$ and $\hat{\mathbf{u}}(t,x)$ provided that $\psi_i(x)$ is positive in Ω for every $i = 1,\dots,N$. The positivity requirement on $\psi_i(x)$ is essential in obtaining a classical solution. Consider, for example, the simple scalar problem

$$\begin{aligned} &\partial u/\partial t - \Delta u^m = 0 && (t>0, \quad x\in\Omega), \\ &u(t,x) = 0 && (t>0, \quad x\in\partial\Omega), \\ &u(0,x) = \psi(x) && (x\in\Omega), \end{aligned}$$

where $m > 1$ and Ω is a bounded domain in $\mathbb{R}^p$ containing the sphere $\{x\in\mathbb{R}^p;\ \|x\| = 1\}$. Since for any $M \geq \psi(x)$ the pair $\tilde{u} = M$ and $\hat{u} = 0$ are ordered upper and lower solutions, this problem has a unique classical solution if $\psi(x) > 0$ in Ω (cf. [28,32]). However, if $\psi(x) = c_0 (1 - |x|)_+^{1/m-1}$ for some positive constant $c_0 \leq 1$ where $(w)_+ = \max\{0, w\}$ then a solution of the above problem is given by

$$u(t,x) = c_0 (t+1)^{\nu} \left((t+1)^{2\mu} - |x|^2 \right)_+^{1/m-1}$$

where

$$\mu = (p(m-1)+2)^{-1}, \quad \nu = \mu(p + 2/(m-1)).$$

Please cite this article in press as: C.V. Pao, W.H. Ruan, Dynamics of degenerate quasilinear reaction diffusion systems with nonnegative initial functions, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.08.024