J. Math. Anal. Appl. 467 (2018) 807–824

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Remarks on the singular set of suitable weak solutions for the three-dimensional Navier–Stokes equations

Wei Ren [a], Yanqing Wang [b,*], Gang Wu [c]

[a] College of Mathematics and Systems Science, Hebei University, Baoding, Hebei 071000, PR China
[b] Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, PR China
[c] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, PR China

ARTICLE INFO

Article history:
Received 14 December 2017
Available online 5 July 2018
Submitted by J. Guermond

Keywords:
Box-counting dimension
Navier–Stokes equation
Suitable weak solution

ABSTRACT

In this study, $\mathcal{S}$ denotes the possible interior singular set of suitable weak solutions for the three-dimensional Navier–Stokes equations. We improve the known upper box-counting dimension of this set from $360/277$ (≈ 1.300) given by [24] to $975/758$ (≈ 1.286). We also show that $\Lambda(\mathcal{S}, r(\log(e/r))^{\sigma}) = 0$ $(0 \leq \sigma < 27/113)$, which extends the previous corresponding results concerning the improvement of the classical Caffarelli–Kohn–Nirenberg theorem by a logarithmic factor given by Choe and Lewis [3, J. Funct. Anal., 175:348–369, 2000], and by Choe and Yang [4, Comm. Math. Phys., 336:171–198, 2015]. The proof is inspired by a new ε-regularity criterion, which was proved by Guevara and Phuc [7, Calc. Var., 56:68, 2017].

1. Introduction

We consider the following incompressible Navier–Stokes equations in three-dimensional (3D) space

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, \quad \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where u denotes the flow velocity field and the scalar function Π represents the pressure. The initial velocity u_0 satisfies $\operatorname{div} u_0 = 0$.

In previous studies, Scheffer in [16–18] proposed a program for estimating the size of the potential space–time singular set $\mathcal{S}$ of (suitable) weak solutions that obey the local energy inequality for the Navier–Stokes system, and proved that the Hausdorff dimension of this set of 3D Navier–Stokes equations is $5/3$ at most. A point is said to be a regular point of the suitable weak solution u provided that the L^{∞} bound of u

* Corresponding author.
E-mail addresses: renwei4321@163.com (W. Ren), wangyanqing20056@gmail.com (Y. Wang), wugangmaths@gmail.com (G. Wu).

https://doi.org/10.1016/j.jmaa.2018.07.003

is in some neighborhood of this point. A point is singular if it is not regular. In addition, the celebrated Caffarelli–Kohn–Nirenberg theorem given by [1] regarding the 3D Navier–Stokes system states that the one-dimensional Hausdorff measure of $\mathcal{S}$ is zero, which was deduced from the following ε-regularity criterion. An absolute constant ε exists such that if

$$\limsup_{r\to 0} r^{-\frac{1}{2}}\|\nabla u\|_{L^2_tL^2_x(Q(r))} \leq \varepsilon, \tag{1.2}$$

then (x, t) is a regular point, where $Q(r) := B(r) \times (t - r^2, t)$ and $B(r)$ denotes the ball of center x and radius r. Subsequently, may studies considered the extension of the Caffarelli–Kohn–Nirenberg theorem and the ε-regularity criteria were presented (e.g., see [2–5,7–15,21–24]).

Recently, by viewing the Bernoulli (total) pressure $\frac{1}{2}|u|^2 + \Pi$ as a signed distribution belonging to a certain fractional Sobolev space of negative order in local energy inequality, Guevara and Phuc in [7] proved the following ε-regularity criterion. If

$$\mu^{-\frac{3}{2}}\left(\left\||u|^2\right\|_{L^p_tL^q_x(Q(\mu))} + \|\Pi\|_{L^p_tL^q_x(Q(\mu))}\right) < \varepsilon_0, \tag{1.3}$$

where (p, q) satisfy

$$2/p + 3/q = 7/2 \text{ with } 1 \leq p \leq 2, \tag{1.4}$$

then (x,t) is a regular point. A particularly interesting case of (1.3) is $p = q = 10/7$, which improves the following classical method given by [11,12] via the blow-up procedure

$$\mu^{-\frac{4}{3}}\left(\left\||u|^2\right\|_{L^{3/2}_tL^{3/2}_x(Q(\mu))} + \|\Pi\|_{L^{3/2}_tL^{3/2}_x(Q(\mu))}\right) < \varepsilon. \tag{1.5}$$

For a pair (p, q) satisfying (1.4), the ε-regularity criterion in terms of the Bernoulli pressure was obtained by [13] as

$$\limsup_{\mu\to 0} \mu^{-\frac{3}{2}}\left(\left\|\frac{1}{2}|u|^2 + \Pi\right\|_{L^p_tL^q_x(Q(\mu))}\right) < \varepsilon.$$

In the present study, we improve the known fractal upper box dimension of $\mathcal{S}$ via (1.3). The relationship between the Hausdorff dimension and the upper box dimension comprises the first being less than the second (e.g., see [6]). The definition of the box dimension is given via the lower box dimension and the upper box dimension. In the following, the box dimension and fractal dimension denote the upper box dimension. First, before we state our theorem, we recall previous related results. Using (1.5), Robinson and Sadowski [14] proved that the upper box dimension of $\mathcal{S}$ is 5/3 at most. Subsequently, Kukavica [9] showed that the box dimension of the singular set is less than or equal to 135/82 (≈1.646) and asked whether the dimension of the singular set is 1 at most. Kukavica and Pei [10] showed that the parabolic fractal dimension of the singular set is less than or equal to 45/29 (≈1.552). Recently, Koh and Yang [8] proved that the fractal upper box dimension of $\mathcal{S}$ is bounded by 95/63 (≈1.508). Based on the arguments given by [8] and some careful estimates, the upper box dimension was refined to 360/277 (≈1.300) by [24].

Our first result comprises the following theorem.

Theorem 1.1. *The (upper) box dimension of* $\mathcal{S}$ *is* 975/758 (≈1.286) *at most.*

Remark 1.1. This improves the previous box dimension of $\mathcal{S}$ obtained by [8–10,14,24].