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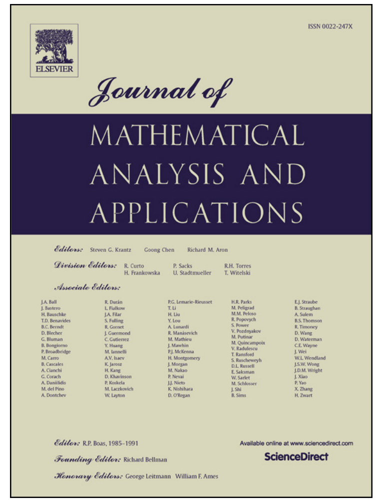

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RECOVERING A PURELY ATOMIC FINITE MEASURE FROM ITS RANGE

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Abstract. Let μ be a purely atomic finite measure. By the range of μ we understand the set $\operatorname{rng}(\mu) = \{\mu(E) : E \subset \mathbb{N}\}$. We are interested in the two following questions. Which set can be a range of some measure μ? Can the purely atomic measure μ be uniquely recovered from its range?

1. Introduction

Assume that μ is a purely atomic finite measure. We may assume that μ is defined on $\mathbb{N}$ and $\mu(\{n\}) \geq \mu(\{n+1\})$. Throughout the paper we assume that measures are always purely atomic, finite and they are defined on $\mathbb{N}$ such that their $n+1$-st atoms have measures not greater than their n-th atoms. We are interested in the following questions:

- For which subsets R of $\mathbb{R}$ there is a measure μ such that R is its range (i.e. $R = \operatorname{rng}(\mu) := \{\mu(E) : E \subset \mathbb{N}\}$)?
- For which subsets R of $\mathbb{R}$ there is exactly one measure μ with $R = \operatorname{rng}(\mu)$?

To simplify the notation let $x_n = \mu(\{n\})$ be a measure of the n-th largest atom of μ. Note that

$$\operatorname{rng}(\mu) = \{\mu(E) : E \subset \mathbb{N}\} = \{\sum_{n\in E} \mu(\{n\}) : E \subset \mathbb{N}\} = \{\sum_{n=1}^{\infty} \varepsilon_n x_n : \varepsilon_n = \{0,1\}^{\mathbb{N}}\}.$$

The latter set is also denoted by $A(x_n)$ and it is called the achievement set of (x_n) (see [16]). Let us present here two simple examples.

Example 1.1. Consider the procedure of rolling dice until the value on the dice is less than 5. For $E \subset \mathbb{N}$ let $\mu_1(E)$ be the probability that the procedure stops for some n from E. Then $\mu_1(\{n\}) = \frac{2}{3^n}$. It is easy to see that for $x_n = \mu_1(\{n\})$ the set $A(x_n)$, or $\operatorname{rng}(\mu_1)$, is equal to the classical Cantor ternary set C.

Example 1.2. Consider the procedure of tossing a fair coin until the head appears. For $E \subset \mathbb{N}$ let $\mu_2(E)$ be the probability that the procedure stops for some n from E. Then $\mu_2(\{n\}) = \frac{1}{2^n}$ and $\operatorname{rng}(\mu_2) = [0,1]$.

Achievement sets of sequences, defined for all summable sequences (x_n), have been considered by many authors; some results have been rediscovered several times. Let us list basic properties of $A(x_n)$ (some of them were observed by Kakeya in [17] in 1914):

(i) $A(x_n)$ is a compact perfect or finite set,
(ii) If $|x_n| > \sum_{i>n} |x_i|$ for all sufficiently large n's, then $A(x_n)$ is homeomorphic to the ternary Cantor set C,
(iii) If $|x_n| \leq \sum_{i>n} |x_i|$ for all sufficiently large n's, then $A(x_n)$ is a finite union of closed intervals. Moreover, if $|x_n| \geq |x_{n+1}|$ for all but finitely many n's and $A(x_n)$ is a finite union of closed intervals, then $|x_n| \leq \sum_{i>n} |x_i|$ for all but finitely many n's.

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