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Isochronicity of a $\mathbb{Z}_2$-equivariant quintic system

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ABSTRACT

The main purpose of this paper is to study the problem of simultaneous existence of two isochronous centers in some families of planar polynomial differential systems with symmetry. More precisely, we present conditions for a family of $\mathbb{Z}_2$-equivariant systems of degree 5 to have an isochronous bi-center. We also study their global phase portraits in the Poincaré disk and present considerations about the number of simultaneous centers in $\mathbb{Z}_2$-equivariant systems of degree 3 and 5. This study provides examples of quintic systems possessing three and five isochronous centers simultaneously.

1. Introduction

A singular point of a planar differential system is called a *center* if all trajectories in its neighborhood are closed curves called periodic orbits, and if all corresponding periodic solutions have the same period we say that the center is *isochronous*. The isochronicity problem, i.e., the problem on the existence of isochronous centers, has been already studied in the 17th century when Huygens investigated the cycloidal pendulum. However, only in the second half of the last century it started to be intensively studied. In [23], Loud investigated the isochronicity of polynomial differential systems of degree two. Latter on, the isochronicity problem was solved for the linear center perturbed by homogeneous polynomials of degree three [25] and degree five [26]. The case of the linear center perturbed by homogeneous polynomials of degree four involves extremely laborious computations and up to now it is still unsolved. Thus, for this case only partial results are obtained, see e.g. [2,6,17]. There are also many papers devoted to the

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investigation of particular families of polynomial differential systems, see e.g. [1,3,4,7,24,29] and references therein.

Regarding to the simultaneous existence of two centers for planar differential equations only very few particular families of systems were studied. Kirnitskaya and Sibirskii in [19] and Li in [20] presented conditions for a planar quadratic differential system to have two centers simultaneously. A particular family of cubic systems was considered by Conti [8]. He studied a cubic system possessing two centers. Also, Chen, Lu and Wang [5] investigated a particular family of cubic systems, the Kukles systems. They characterized when such systems possess two centers simultaneously.

In this paper we are interested in a class of systems with $\mathbb{Z}_2$-equivariant symmetry, i.e., systems with unchanged phase portraits after a rotation on the angle π around a point P (for more details about $\mathbb{Z}_q$-equivariant systems and their importance for studies on Hilbert's 16th problem, see [18,21]). Assume that A and B are singular points of a differential system which is $\mathbb{Z}_2$-equivariant with respect to the middle point of the line segment AB. We say that such system has a *bi-center* at points A and B if both A and B are singular points of the center type.

Recently Liu and Li [22] studied a planar $\mathbb{Z}_2$-equivariant cubic system. They presented necessary and sufficient conditions for the existence of a bi-center at the points $(-1, 0)$ and $(1, 0)$. Later on, Du and Liu [13] and Romanovski, Fernandes and Oliveira [27] characterized the isochronicity of the centers obtained in [22]. In [12] Du investigated the existence of two centers and their isochronicity for a particular family of $\mathbb{Z}_2$-equivariant system of degree seven. Giné, Llibre and Valls, [18], provided conditions for the existence of multiple centers for $\mathbb{Z}_2$-equivariant cubic and quintic systems. They obtained conditions for the simultaneous existence of a center at the origin and two more centers at the points (a, b) and $(-a, -b)$ with $ab \neq 0$ arbitrary.

In [27], the authors also investigated the existence of a bi-center for a subfamily of a planar $\mathbb{Z}_2$-equivariant differential quintic system of the form

$$\dot{x} = X_1(x, y) + X_5(x, y) = X(x, y), \qquad \dot{y} = Y_1(x, y) + Y_5(x, y) = Y(x, y), \tag{1.1}$$

where $X_i(x, y)$, $Y_i(x, y)$ $(i = 1, 5)$ are homogeneous polynomials of degree i in the variables x and y and (1.1) has two weak foci or centers at the points $(-1, 0)$ and $(1, 0)$. Four families of system (1.1) possessing bi-centers were found and the authors also have shown that they are not isochronous.

The main propose of this paper is to continue the investigation of system (1.1) studying conditions on the parameters for such system to have an isochronous bi-center. The main results of this paper are described in the sequence. Theorems 4.1 and 4.2 characterize the existence of an isochronous bi-center for a subfamily of system (1.1). Theorem 5.1 provides the global dynamics of the systems obtained in Theorem 4.1. Theorem 6.1 studies the coexistence of more than two isochronous centers for a subfamily of system (1.1). From this study we obtain examples of quintic systems possessing 3 and 5 isochronous centers simultaneously.

The paper is organized as follows. In Sections 2 and 3 we present some definitions and results about the isochronicity problem and Poincaré compactification. In Section 4 we state and prove Theorems 4.1 and 4.2. In Section 5 we prove Theorem 5.1 and in Section 6 we present and prove Theorem 6.1.

2. Linearizability quantities and Darboux linearization

In this section we remind some results related to isochronicity and linearizability problems for polynomial differential systems of the form

$$\dot{x} = -y + \sum_{p+q=2}^{n} a_{p,q} x^p y^q = P(x, y), \qquad \dot{y} = x + \sum_{p+q=2}^{n} b_{p,q} x^p y^q = Q(x, y), \tag{2.1}$$