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Instability of set differential equations

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ABSTRACT

This paper is devoted to the instability of Set Differential Equations (SDEs). Using the geometric inequalities of Brunn–Minkowski and A.D. Aleksandrov, we propose new methods for constructing Lyapunov functions. In combination with the known methods of stability theory, the Lyapunov's direct method, the comparison method and the vector-function method, we establish conditions for the collapse of the solutions of the SDEs. Estimates of the collapse time of solutions are also obtained. Examples of SDEs in spaces of dimension 2 and 3 illustrating general theorems are given.

0. Introduction

Differential equations with Hukuhara derivative (Set Differential Equations (SDEs)) were first considered in [3]. Further development of the theory of differential equations with Hukuhara derivative has been summarized in the monograph [5], where the conditions of existence and uniqueness of solutions of the Cauchy problem, the convergence of successive approximations including the principle of comparison and the theorems of Lyapunov's direct method have been formulated.

We note that in the definition of the concepts of stability and the asymptotic stability of the solutions of SDEs, certain difficulties arise, connected with the fact that the function is non-decreasing. The problem of determining the concepts of stability and asymptotic stability for SDEs is not a trivial subject. Some approaches to solving this problem are proposed in [12]. Despite the fact that the property of stability and asymptotic stability is not typical for SDEs, many papers [6,7,13] are devoted to the study of these properties of solutions. Much less work has been devoted to the study of instability for SDEs. The purpose of this paper is to investigate the instability of the SDEs. More precisely, here we investigate the collapse

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conditions for the solutions of SDEs. By collapse of the solution, we mean the solution tends to infinity in a finite time.

In papers [10,11], the results and methods of geometry of convex bodies, which are developed in the classical works of H. Minkowski and A.D. Alexandrov [1,2,9], were used for the study of stability of solutions for dynamical systems in the space of convex compact sets in $\mathbb{R}^n$. In this paper, these ideas are applied to the study of instability of solutions for SDEs.

For this purpose, it is proposed to use some mixed volume functionals as the scalar Lyapunov function and as components of the Lyapunov vector function. In the special case of spaces of dimensions 2 and 3, these functionals have a simple geometric meaning.

1. Auxiliary results

Let conv $\mathbb{R}^n$ be the metric space of convex compacts sets from $\mathbb{R}^n$ with Hausdorff metric. In conv $\mathbb{R}^n$ the operations of addition (addition of Minkowski) and multiplication by a nonnegative scalar are defined. If $A \in L(\mathbb{R}^n)$, where $L(\mathbb{R}^n)$ is Banach algebra of linear operators $\mathbb{R}^n$, then the action of the operator A can be extended in a natural way to the space conv $\mathbb{R}^n$,

$$AX = \{Ax : x \in X\} \in \operatorname{conv} \mathbb{R}^n, \quad X \in \operatorname{conv} \mathbb{R}^n.$$

Let $X, Y \in \operatorname{conv} \mathbb{R}^n$. If there exists an element $Z \in \operatorname{conv} \mathbb{R}^n$ such that $X = Z + Y$, then element Z is called the Hukuhara difference of the elements X and Y, and it is denoted as $Z = X - Y$. The difference of two elements of the space conv $\mathbb{R}^n$ does not always exist. If X and Y are two nonempty subsets of $\mathbb{R}^n$, then the distance (Hausdorff metric) between them is defined by

$$d_H[X,Y] = \inf_{\varepsilon \geq 0} \{X \subset Y + \varepsilon K, \quad Y \subset X + \varepsilon K\},$$

where K is a closed unit ball in $\mathbb{R}^n$. Let $\Theta = \{0\}$. We denote $\|X\| = d_H[X, \Theta]$.

The notion of Hukuhara difference allows us to define the notion of Hukuhara derivative by the mapping $F : (\alpha, \beta) \to \operatorname{conv} \mathbb{R}^n$, $(\alpha, \beta) \subset \mathbb{R}$.

Definition 1. [5] A mapping $F : (\alpha, \beta) \to \operatorname{conv} \mathbb{R}^n$ is said to be differentiable at a point $t_0 \in (\alpha, \beta)$, if there exists an element $D_H F(t_0) \in \operatorname{conv} \mathbb{R}^n$ such that the limits

$$\lim_{\varrho \to 0+} \frac{F(t_0 + \varrho) - F(t_0)}{\varrho}, \quad \lim_{\varrho \to 0+} \frac{F(t_0) - F(t_0 - \varrho)}{\varrho}$$

exist and are $D_H F(t_0)$.

In this case, $D_H F(t_0)$ is called the Hukuhara derivative at the point t_0.

Note that the necessary condition for the differentiability of the mapping is that the function diam $F(t)$ is nondecreasing.

Let $V[X]$ be the volume of the body X and $X_k \in \operatorname{conv} \mathbb{R}^n$, $\lambda_k \in \mathbb{R}$ and $\lambda_k \geq 0$, $k = 1, \ldots, m$, $X = \sum_{k=1}^{m} \lambda_k X_k \in \operatorname{conv} \mathbb{R}^n$. H. Minkowski has shown that a volume $V[X]$ of a convex body X is a homogeneous polynomial of degree n relative to the variables λ_k:

$$V[X] = \sum_{k_1, \ldots, k_n} \lambda_{k_1} \ldots \lambda_{k_n} V_{k_1, k_2, \ldots, k_n}, \tag{1.1}$$

where the sum is taken over all indices $k_1, \ldots, k_n$ which vary independently over all values from 1 to m. At the same time, the coefficients of $V_{k_1, \ldots, k_n}$ are determined so that they do not depend on the order of the

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