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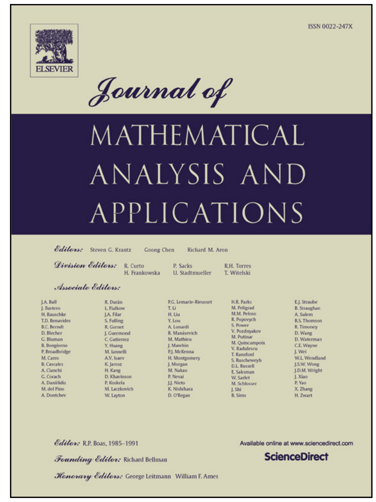

Fixed points of polarity type operators

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FIXED POINTS OF POLARITY TYPE OPERATORS

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Abstract. A well-known result says that the Euclidean unit ball is the unique fixed point of the polarity operator. This result implies that if, in $\mathbb{R}^n$, the unit ball of some norm is equal to the unit ball of the dual norm, then the norm must be Euclidean. Motivated by these results and by relatively recent results in convex analysis and convex geometry regarding various properties of order reversing operators, we consider, in a real Hilbert space setting, a more general fixed point equation in which the polarity operator is composed with a continuous invertible linear operator. We show that if the linear operator is positive definite, then the considered equation is uniquely solvable by an ellipsoid. Otherwise, the equation can have several (possibly infinitely many) solutions or no solution at all. Our analysis yields a few by-products of possible independent interest, among them results related to coercive bilinear forms (essentially a quantitative convex analytic converse to the celebrated Lax-Milgram theorem from partial differential equations) and a characterization of real Hilbertian spaces.

1. Introduction

1.1. **Background:** Consider the following geometric fixed point equation:

$$C = (GC)^\circ. \tag{1.1}$$

Here $C \neq \emptyset$ is the unknown subset which is assumed to be contained in a given real Hilbert space $X \neq \{0\}$, $G : X \to X$ is a given continuous, invertible and linear operator, $GC := \{Gc : c \in C\}$, and S° denotes the polar (or dual) of $\emptyset \neq S \subseteq X$ (see **(2.6)** below).

In this paper we analyze and solve **(1.1)** under various assumptions on C and on G. The motivation to consider **(1.1)** is based on a number of reasons. First, **(1.1)** is a generalization of the equation

$$C = C^\circ, \tag{1.2}$$

which describes all the self-polar sets. A well-known and classical result in convex geometry says that there exists a unique self-polar set and this set is the unit ball (see, for example, [**6**, p. 138], [**7**, pp. 144–145], [**17**, p. 148]). This result implies that if we start with $\mathbb{R}^n$ and want to define on it a norm such that the unit ball induced by this norm coincides with the unit ball of the dual norm, then we can

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