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Free boundary problem of a reaction–diffusion equation with nonlinear convection term

Ren-Hu Wang [a,b], Lei Wang [a], Zhi-Cheng Wang [a,*]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China

[b] School of Mathematics and Statistics, Hexi University, Zhangye, Gansu 734000, People's Republic of China

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ABSTRACT

In this paper, we consider the free boundary problem of a reaction diffusion equation with nonlinear convection term in one dimensional space. Our study contains three parts: in the first part we establish the existence and uniqueness of global solution, in the second part we obtain the spreading–vanishing dichotomy, and in third part, we obtain some estimations of the asymptotic speed of free boundaries when spreading happens.

1. Introduction

Nonlinear diffusion problems with free boundary conditions are generally used to describe the expansion and propagation of biological species or chemical substances, and the free boundary is used to represent the frontier of this expansion. It has been a central issue in ecological research to explore the law of population expansion of new species or invasive species in new environment. A large amount of empirical evidence shows that many invasive species, which survived in new environment, will expand at a fixed rate after a very short initial stage. A classic example is the rules discovered by Skellam [19] in 1951, which describes the spreading of muskrat in Europe in early twentieth century.

One of the most successful mathematical descriptions of the propagation of species is based on the theory of traveling wave solutions. In 1937, Fisher [11] made use of the equation

$$u_t - du_{xx} = au - bu^2, \quad t > 0, \ x \in \mathbb{R}^1, \tag{1.1}$$

* Corresponding author.
E-mail address: wangzhch@lzu.edu.cn (Z.-C. Wang).

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to study the transmission pattern of advantageous genes, where the function u represents the population density of species that carries advantageous genes. He proved that equation (1.1) admits a solution of the form $u(t,x) = \omega(x - ct)$, if $c \geq 2\sqrt{ad} := c^*$ and ω satisfies

$$c\omega' + d\omega'' + a\omega - b\omega^2 = 0, \quad \omega(-\infty) = 0, \quad \omega(+\infty) = a/b.$$

He also showed that there exists no such solutions if $c < 2\sqrt{ad}$. Fisher claimed that c^* is the spreading speed for the advantageous genes in his research. The same results were proved by Kolmogorov et al. for a more general class of equations whose nonlinearity is now called Fisher-KPP type. In 1975, Aronson and Weinberger [1] established a more general theory based on the traveling wave solutions. They proved that for any $\varepsilon > 0$, the solution of (1.1) satisfies:

$$\lim_{t\to\infty, |x|\leq(c^*-\varepsilon t)} u(t,x) = a/b, \qquad \lim_{t\to\infty, |x|\geq(c^*+\varepsilon t)} u(t,x) = 0.$$

This means that if an observer travels in the direction of propagation at a speed c which is smaller than c^*, then it will find that the population is close to $\frac{a}{b}$, and if his speed is bigger than c^*, it would observe that the population is nearly 0. The mathematical results have been extended to higher dimensions in [2] by Aronson and Weinberger.

The mathematical model mentioned above have obtained lots of good results, but the reaction–diffusion equation, which were used in these models to describe the expansion behavior, will force infinite propagation speed. Namely, for any initial population distribution which is nonnegative and is positive somewhere, for $t > 0$ the population density at any location is bigger than 0, which does not conform with the reality. To overcome the difficulty, Du and Lin [7] first began to try to introduce free boundary conditions to study the expansion behavior of biological populations, and they considered the following equations:

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \leq x \leq h_0, \end{cases} \tag{1.2}$$

where $h(t)$ stands for the free boundary, the function u represents population density. The free boundary $h(t)$ satisfies equation $h'(t) = -\mu u_x(t,h(t))$, which is a special case of the well-known Stefan condition and the deduction can refer to [31]. If the right side is free boundary, the left side is fixed boundary condition, then the equation (1.2) describes the expansion behavior of new species or invasive species populations in one dimensional environment, they got the following results:

(1) Equation (1.2) has a unique global solution;
(2) Spreading–vanishing dichotomy and its criterion: if $t \to \infty$, then either $h(t) \to \infty$ and $u(t,x) \to \frac{a}{b}$, which we called spreading, or $h(t) \to h_\infty \leq \frac{\pi}{2}\sqrt{d/a}$ and $u(t,x) \to 0$, which we called vanishing. Furthermore, there exists a positive constant μ^* that describes the ability of expansion, if $\mu > \mu^*$, then vanishing happens; if $0 \leq \mu \leq \mu^*$, then spreading happens, and μ^* depends on u_0 and h_0.
(3) Some basic estimates for the asymptotic spreading speed of two fronts when spreading happens, which means that there exists a positive constant k_0 that only depends on μ, and

$$\lim_{t\to\infty} \frac{h(t)}{t} = k_0, \qquad \lim_{t\to\infty} -\frac{g(t)}{t} = k_0.$$

After this pioneering work, a more general reaction diffusion problem with free boundary conditions was considered by Du and Lou [9]: