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Rational dilation problems associated with constrained algebras ☆

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A B S T R A C T

A set Ω is a spectral set for an operator T if the spectrum of T is contained in Ω, and von Neumann's inequality holds for T with respect to the algebra $R(\Omega)$ of rational functions with poles off of $\overline{\Omega}$. It is a complete spectral set if for all $r \in \mathbb{N}$, the same is true for $M_r(\mathbb{C}) \otimes R(\Omega)$. The rational dilation problem asks, if Ω is a spectral set for T, is it a complete spectral set for T? There are natural multivariable versions of this. There are a few cases where rational dilation is known to hold (eg, over the disk and bidisk), and some where it is known to fail, for example over the Neil parabola, a distinguished variety in the bidisk. The Neil parabola is naturally associated to a constrained subalgebra of the disk algebra $\mathbb{C} + z^2 A(\mathbb{D})$. Here it is shown that such a result is generic for a large class of varieties associated to constrained algebras. This is accomplished in part by finding a minimal set of test functions. In addition, an Agler–Pick interpolation theorem is given and it is proved that there exist Kaijser–Varopoulos style examples of non-contractive unital representations where the generators are contractions.

1. Introduction

It was first recognized in the 1950s that there is a deep connection between the fact that over the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, von Neumann's inequality holds for any Hilbert space contraction operator, and that a contraction can be dilated to unitary operator (the Sz.-Nagy dilation theorem). A similar phenomenon is observed for a commuting pair of contractions, which according to Andô's theorem, dilate to a commuting pair of unitary operators.

More generally, an operator T in $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$, is said to have a *rational dilation* (with respect to a compact set $\overline{\Omega}$) if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ with spectrum in the boundary of $\overline{\Omega}$ such that $f(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}$ for all $f \in R(\overline{\Omega})$, the rational functions with poles off of $\overline{\Omega}$.

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There is a natural multivariable version of this.

Problem *(Rational dilation problem*[1]*).* Let Ω be a domain in $\mathbb{C}^n$ with compact closure and suppose that T is a commuting tuple of bounded operators on a Hilbert space $\mathcal{H}$ with spectrum contained in $\overline{\Omega}$. Furthermore, assume that for every $f \in R(\Omega)$, the set of rational functions with poles off of $\overline{\Omega}$, the von Neumann inequality holds; that is, $\|f(T)\| \leq \|f\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm over $\overline{\Omega}$. Does there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and commuting tuple of normal operators N on $\mathcal{K}$ with spectrum in the boundary of Ω such that $f(T) = P_{\mathcal{H}} r(N)|_{\mathcal{H}}$ for all $f \in R(\overline{\Omega})$? That is, does T have a *rational dilation* to N?

Here Arveson [6] is followed in defining the spectrum of a tuple T to be $\sigma(T) := \{\lambda \in \mathbb{C}^n : \text{ for } p : \mathbb{C}^n \to \mathbb{C} \text{ a polynomial, } p(\lambda) \in \sigma(p(T))\}$. He showed that this set is non-empty and compact, and that the spectral mapping theorem holds for all non-constant rational functions with poles off of $\sigma(T)$.

When the von Neumann inequality holds for an operator (or tuple of operators) T as in the statement of the rational dilation problem, $\overline{\Omega}$ is said to be a *spectral set* for T. It is not difficult to see that if T has a rational dilation, then $\overline{\Omega}$ is a spectral set for T; indeed, one also has the von Neumann inequality for $f \in R(\Omega) \otimes M_r(\mathbb{C})$, the matrix valued rational functions with poles off of $\overline{\Omega}$, for any finite r. Hence $\overline{\Omega}$ is a *complete spectral set* for T.

A nontrivial fact, also due to Arveson [6], is that T has a rational dilation if and only if $\overline{\Omega}$ is a complete spectral set for T. Thus the rational dilation problem can be reformulated as: *If $\overline{\Omega}$ is a spectral set for T, is it a complete spectral set for T?*

Given a set $X \subset \mathbb{C}^d$, a function $f : X \to \mathbb{C}$ is *analytic* if for every $x \in X$, there is an open neighborhood of x to which f extends analytically. Denote by $A(\Omega)$ the subalgebra of functions in $C(\overline{\Omega})$ which are analytic on Ω. At least over subsets of $\mathbb{C}$, there are various conditions which imply that $R(\Omega)$ is dense in $A(\Omega)$; for example, if Ω is finitely connected, then this is true. In this paper we concentrate on the setting where Ω is the intersection of a variety with $\overline{\mathbb{D}}^n$. Since the variety is the zero set of a polynomial, similar reasoning as in the one variable setting will dictate that $R(\Omega)$ is dense in $A(\Omega)$. How a bounded representation acts on $R(\Omega)$ is determined by its action on the generators, so such a representation extends continuously to $A(\Omega)$. This gives yet another formulation of the rational dilation problem over suitably nice Ω: *Is every contractive representation of $A(\Omega)$ completely contractive?*

An implication of the Sz.-Nagy dilation theorem is that contractive representations of $A(\mathbb{D})$ are completely contractive, and Andô's theorem allows us to draw the same conclusion for $A(\mathbb{D}^2)$. So for $\Omega = \mathbb{D}$ or $\mathbb{D}^2$, rational dilation holds. A more substantial argument is needed to prove that rational dilation holds for annuli [1] (but see also [17]), and intriguingly, there is a way of mapping an annulus to a distinguished variety of the bidisk [28] (that is, a variety $\mathcal{V}$ which intersects $\mathbb{D}^2$ and satisfies $\mathcal{V} \cap \partial\mathbb{D}^2 \subset \mathbb{T}^2$, which is the *distinguished*, or *Šilov boundary* of $\mathbb{D}^2$). Thus rational dilation holding for annuli is equivalent to it holding for a certain family of distinguished varieties in $\mathbb{D}^2$. It is natural to wonder if this is a legacy of rational dilation holding over $\mathbb{D}^2$, and so to speculate that perhaps rational dilation also holds for other distinguished varieties in $\mathbb{D}^2$.

Alas, this is too much to hope for. In [17], it was proved that rational dilation fails for the Neil parabola $\mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$. The techniques are indirect. As with an annulus, one can associate $A(\mathcal{N})$ to another algebra. In this case, there is a complete isometry mapping $A(\mathcal{N})$ onto $A_{z^2}(\mathbb{D}) = \mathbb{C} + z^2 A(\mathbb{D})$, the subalgebra of $A(\mathbb{D})$, the functions of which have first derivative vanishing at 0. It is shown in [17] that this algebra has a contractive representation which is not 2-contractive, and so not completely contractive.

In this paper, we show that rational dilation fails without fail for algebras $A(\mathcal{V}_B)$ of functions which are analytic and continuous up to the boundary on distinguished varieties $\mathcal{V}_B$ of the N-disk associated to finite Blaschke products B with $N \geq 2$ zeros. We also prove that it fails on associated distinguished varieties of

[1] This problem is usually attributed to Halmos, and while this seems plausible, we have been unable to find a reference!