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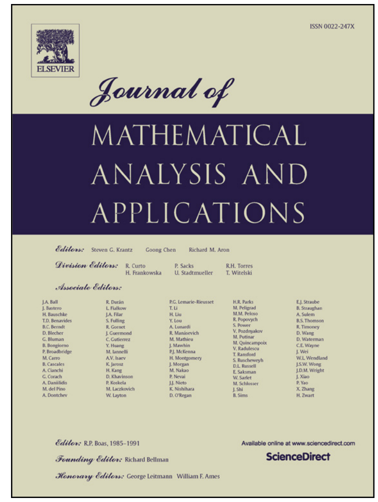

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The diffusion equation with nonlocal data

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Abstract

We study the diffusion (or heat) equation on a finite 1-dimensional spatial domain, but we replace one of the boundary conditions with a "nonlocal condition", through which we specify a weighted average of the solution over the spatial interval. We provide conditions on the regularity of both the data and weight for the problem to admit a unique solution, and also provide a solution representation in terms of contour integrals. The solution and well-posedness results rely upon an extension of the Fokas (or unified) transform method to initial-nonlocal value problems for linear equations; the necessary extensions are described in detail. Despite arising naturally from the Fokas transform method, the uniqueness argument appears to be novel even for initial-boundary value problems.

1 Introduction

Consider the apparatus arranged as in figure 1. A clear tube contains a colloidal suspension whose opacity is a known monotonic function of the concentration of the dispersed substance. At $x = 1$, the tube is terminated so that the flux of dispersed substance across the boundary is zero. Assuming no net flow of the liquid phase, no variation in viscosity or temperature, and no external agitation, the dispersed substance diffuses according to the 1-dimensional heat equation. Therefore, assuming the initial concentration profile, q_0, is known, a measurement, γ, of the concentration of dispersed substance at position $x = 0$ for all time $t > 0$ specifies the well-posed initial-boundary value problem

$$\begin{aligned}
[\partial_t - \partial_x^2]q(x,t) &= 0 & (x,t) &\in (0,1)\times(0,T),\\
q(x,0) &= q_0(x) & x &\in [0,1],\\
q_x(1,t) &= 0 & t &\in [0,T],\\
q(0,t) &= \gamma(t) & t &\in [0,T],
\end{aligned}$$

which may be solved, via a classical Fourier series or Green's function approach, for the concentration $q(x,t)$ at any interior point.

One must consider how a measurement of γ could practically be made. One approach is to use a lamp (or laser) and photovoltaic cell to measure the opacity of the colloidal suspension, thereby to deduce the

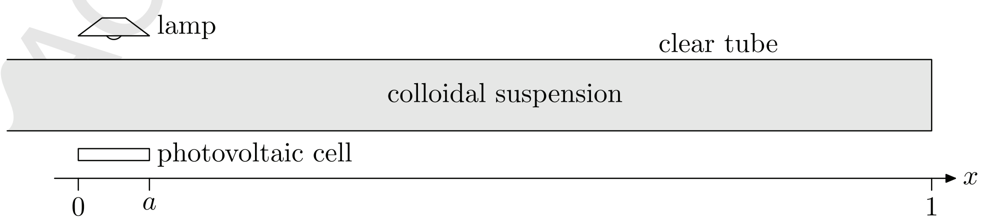

Figure 1: Measurement of concentration of dispersed substance in colloidal suspension